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THE UNIVERSITY OF ALBERTA

AXIALLY SYMMETRIC PROBLEMS IN PLASTICITY

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DIVISION OF APPLIED MECHANICS

EDMONTON, ALBERTA

SEPTEMBER, 1962

ABSTRACT

This thesis is a study of quasi-static, axially symmetric plastic deformation of a rigid-plastic, non-hardening material which obeys the Tresca yield criterion. Investigations of the various plastic regimes possible as the stress point traverses the Tresca yield locus are undertaken and it is shown that, in all non-trivial cases, the equations governing the associated stress and velocity fields of each plastic regime are hyperbolic. Particular attention is given to the regime representative of the Haar-Kármán hypothesis. In view of the success of this hypothesis in the solution of many problems in axially symmetric indentation, it has been used in the thesis with regard to an indentation problem with a cone.

A rigid-plastic, non-hardening material, which is semi-infinite in extent, is indented by rigid, smooth, right circular conical punch initially inserted into a prepared cavity at the surface of the material. By using the plastic stress field of the "incomplete" solution, a value of $4.6424\pi kR^2$ is obtained for the yield point load, where k is the maximum shearing stress for the material and R is the surface radius of the prepared cavity.

ACKNOWLEDGEMENTS

The author wishes to extend his appreciation to Dr. J. B. Haddow for his guidance and supervision of this thesis.

Additional thanks are extended also to those members of the University of Alberta Computing Centre whose assistance proved quite helpful.

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CHAPTER I

INTRODUCTION

1.1 Historical Survey.

The work of Tresca on the flow of metals during punching and extrusion is considered the beginning of the mathematical theory of plasticity. Tresca [1864] advanced the theory that a metal yields plastically when the maximum shearing stress attains a critical value. This hypothesis is now called the Tresca yield criterion. Saint-Vénant [1870] used this criterion to interpret the results of experiments on the torsion and bending of cylinders. Saint-Vénant proposed that during two dimensional plastic flow in isotropic materials a co-axial relationship exists between the stress tensor and the strain-rate tensor. Lévy [1870] extended this idea of Saint-Vénant's to three dimensions and postulated proportionality of the stress deviator components and the corresponding plastic strain-rate components. Von Mises [1913] suggested the same proportionality relationship (not knowing of Lévy's earlier work) and postulated a new criterion for the yielding of metals. The Mises yield criterion implies that yielding of a material begins when the octahedral shearing stress reaches a critical value. Von Mises extended his work in 1928 to perfectly plastic solids having a general yield function. He derived the plastic stress-strain relations corresponding to this yield function and from this deduced the concept of the plastic potential and its associated flow rule. Through the independent works of Melan [1938] and Prager [1949], the general plastic stress-strain relations for solids with a regular yield function were obtained. Meanwhile, Reuss [1933] had investigated the flow rule associated with the Tresca singular yield criterion. Koiter [1953a] adapted the theory of plastic potential to

materials with singular yield functions and this resulted in the Koiter-Prager generalization of the Mises theory of plastic potential.

Application of the mathematical techniques associated with the plastic potential has resulted in the solution of many problems in plane strain plasticity. However, very few, non-trivial, axially symmetric problems in plasticity have been solved. Investigators such as Hill [1948], Symonds [1949] and Parsons [1956] have shown that when the Mises yield criterion and its associated flow rule are used in axially symmetric problems, the plastic stress and velocity equations are not hyperbolic. Serious mathematical difficulties arise in finding solutions to the problems under such conditions. Koiter [1953b] used the Tresca yield criterion and associated flow rule to obtain closed solutions for many problems of partially plastic, thick-walled tubes under axial end-conditions. Shield [1955] solved the axially symmetric problem of incipient plastic flow of a semi-infinite, non-hardening, rigid-plastic material indented by a rigid, smooth, cylindrical indenter. In his analysis, Shield used the Tresca yield criterion and its associated flow rule together with the Haar-von Kármán hypothesis. This hypothesis had been under criticism by many authorities of plasticity; e.g., Hill [1950a], since its inception in 1909. Haar and von Kármán [1909] postulated that in some statically determinate problems of axial symmetry, the circumferential stress is equal to one of the principal stresses in the axial plane. Application of this hypothesis without justification was the main criticism. Ishlinskii [1944] attempted the problem of indentation of the plane surface of a semi-infinite material by a circular, flat-ended, smooth, rigid punch. Berezancev [1955] attempted the problem in soil mechanics of normal penetration of cohesive soils by a rigid, smooth, right circular cone. Both Ishlinskii and Berezancev assumed

the Haar-von Kármán hypothesis. The criticism directed to Ishlinskii's work, other than the ad hoc use of the Haar-von Kármán hypothesis, was the inherent inaccuracy of the graphical method used in obtaining the plastic stress field and in not attempting to find the associated plastic velocity field. Berezancev's work is only approximate due to his assumed conditions on the boundary. Also, there was no attempt made to find an associated velocity field. It was not until Shield [1955] had derived the exact solution of Ishlinskii's problem that these works were considered justified in utilizing the Haar-von Kármán hypothesis.

1.2 Scope of Thesis.

In this thesis, the Koiter-Prager generalization of the von Mises theory of plastic potential is used to derive the quasi-static stress and velocity equations for a material deforming plastically under conditions of axial symmetry. The material obeys the Tresca yield criterion and is assumed to be rigid-plastic, non-hardening and isotropic in nature. The various plastic regimes possible as the stress point traverses the Tresca yield locus are conveniently represented as members of four distinct groups. In each group, the field equations for the associated plastic regimes are shown to be non-elliptic.

The hyperbolic stress equations of the plastic regime represented by the Haar-von Kármán hypothesis are used in an indentation problem with a cone. A material, semi-infinite in extent, rigid-plastic and non-hardening, has on its stress free plane surface a right conical cavity whose axis is normal to the plane surface. A rigid, smooth, right circular conical indenter is inserted into this cavity occupying it fully. The indenter is then normally loaded until plastic flow of the material occurs.

The plastic stress field and the yield point load for incipient plastic flow of the material is determined by methods of numerical analysis. No attempt was made to derive an associated kinematically admissible velocity field or to extend the stress field into the remaining rigid region.

CHAPTER II

GENERAL FIELD EQUATIONS FOR PLASTIC FLOW UNDER QUASI-STATIC AXIALLY SYMMETRIC CONDITIONS.

2.1 Conditions of Axial Symmetry.

Let O be the origin of a right-handed 3-dimensional system of cylindrical polar coordinates r, θ, z (Fig. 1). With respect to this system, the physical components of the stress tensor are denoted as $(\sigma_r, \sigma_\theta, \sigma_z, \sigma_{\theta z}, \sigma_{rz}, \sigma_{r\theta})$, the physical components of the strain-rate tensor as $(\epsilon_r, \epsilon_\theta, \epsilon_z, \epsilon_{\theta z}, \epsilon_{rz}, \epsilon_{r\theta})$, and the physical components of the velocity as (u, v, w).

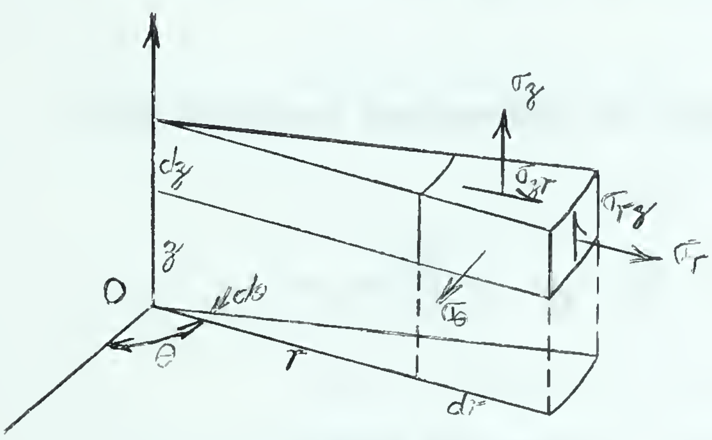


Figure 1
Cylindrical Polar Coordinate System and Stress Components for Axially Symmetric Conditions.

Under conditions of axial symmetry, the choice of reference plane is arbitrarily any meridian plane. The shear components $\sigma_{\theta z}$ and $\sigma_{r\theta}$, the velocity component v, and the shear strain-rate components $\epsilon_{\theta z}$ and $\epsilon_{r\theta}$ must be zero. The remaining components of stress, velocity and strain-rate are then expressible as functions of r, z and T where T is physical time. However, in any quasi-static problem, a time-scale is necessary only to order events in contrast to dynamical problems where inertial effects are considered and the physical time must be used. Thus in quasi-static problems, any suitable monotonically increasing parameter t correlated with progressive deformation may be used as a time-scale.

If body forces are neglected and quasi-static conditions are assumed, the equations of equilibrium satisfied by the stress components are

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2.1.1}$$

and

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \; . \tag{2.1.2}$$

The physical components of strain-rate are expressible as

$$\epsilon_r = \frac{\partial u}{\partial r} , \quad \epsilon_\theta = \frac{u}{r} , \quad \epsilon_z = \frac{\partial w}{\partial z} , \quad \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) . \tag{2.1.3}$$

Along the axis, $r = 0$, conditions are imposed upon the components of stress, velocity, strain-rate and their respective derivatives. Derivation of these conditions are based on the assumption that plastic flow occurs without fracture. Also, conditions must be such as to insure the existence of derivatives and certain limits of stress, velocity and strain-rate upon approaching the z-axis. From the equations of equilibrium (2.1.1) and (2.1.2), it cannot be concluded that $\sigma_r - \sigma_\theta = \sigma_{rz} = 0$ when $r = 0$ in order to avoid infinite values of the stresses. Although stresses may be bounded, there is no reason why their derivatives should not become unbounded on approaching the axis. σ_r , at any point P, is defined as the normal stress on a plane through P perpendicular to a radius through P; and σ_θ is defined as the normal stress, at P, on a plane through P containing the z-axis. Hence, as argued by Parsons [1956], these definitions must hold for all points and, in particular, for those on the z-axis. When P lies on this axis, any plane through P containing the axis is also perpendicular to a radius drawn through P. Hence σ_r and σ_θ at points on the axis

are the same stress and $\sigma_r - \sigma_\theta = 0$. Also for any point P on or off the z-axis, σ_{zr} is defined as the tangential stress at P on a plane through P perpendicular to the z-axis. If P lies on this axis, it follows that $\sigma_{zr} = 0$, for if $\sigma_{zr} \neq 0$ this tangential stress on the plane would have a definite direction at P, contradicting the hypothesis of axial symmetry. Hence, conditions of axial symmetry and proportionality of the components of the stress and plastic strain-rate tensor require that

$$\left.\begin{aligned} \sigma_r = \sigma_\theta \,,\ \sigma_{zr} = 0 \\ \epsilon_r = \epsilon_\theta \,,\ \epsilon_{zr} = 0 \end{aligned}\right\} \text{ on } r = 0 \,. \qquad (2.1.4)$$

The condition that particles near the axis, $r = 0$, do not separate requires that $u = 0$ when $r = 0$.

From (2.1.1), it follows that

$$\lim_{r \to 0} \left(\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} \right) = 0 \,.$$

But since $\sigma_{rz} = 0$ on $r = 0$, $\lim\limits_{r \to 0} \dfrac{\partial \sigma_{rz}}{\partial z} = 0$. Hence

$$\lim_{r \to 0} \left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} \right) = 0 \,. \qquad (2.1.5)$$

From (2.1.2), it also follows that

$$\lim_{r \to 0} \left(\frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} \right) = - \lim_{r \to 0} \frac{\partial \sigma_{rz}}{\partial r}$$

$$= - \lim_{r \to 0} \frac{\sigma_{rz}}{r} \,.$$

Hence

$$\lim_{r \to 0} \left(\frac{\partial \sigma_z}{\partial z} + \frac{2\sigma_{rz}}{r} \right) = 0 \quad . \tag{2.1.6}$$

From (2.1.4), $\epsilon_{rz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) = 0$ on $r = 0$.

Since $u = 0$ on $r = 0$, $\frac{\partial u}{\partial z} = 0$ on $r = 0$.

Hence
$$\lim_{r \to 0} \frac{\partial w}{\partial r} = 0 . \tag{2.1.7}$$

Condition (2.1.7) insures continuity of ϵ_{rz} at the axis of symmetry. (2.1.15),(2.1.6),(2.1.7) are the conditions on the limits of the respective quantities which must be satisfied under axial symmetry.

2.2 Physical Conditions of Yield and Plastic Flow.

Consideration is now confined to a rigid-plastic, non-hardening, homogeneous and isotropic material which obeys the Tresca yield criterion. Rigid-plastic implies incompressibility and the vanishing of all elastic strains. Non-hardening expresses the condition that non-vanishing plastic strain-rates may occur when the material is under a constant state of stress at the yield limit.

The Tresca yield criterion of constant maximum shearing stress for isotropic materials is conveniently described in a 3-dimensional principal stress space. In this space, the principal stress components σ_1, σ_2, σ_3 are chosen as rectangular Cartesian coordinates and any state of stress is represented by a point σ_i $(i = 1,2,3)$. The Tresca yield criterion is represented by the surface of a regular hexagonal prism with its axis equally inclined to the positive σ_1, σ_2, σ_3 axes and passing through the origin. No change in the plastic strains of an element of a material occurs if the stress point lies within the prism. Such states are referred to as "safe". Increments of plastic strain can occur only if the

stress point is on the surface; hence, the surface is called the yield surface. States of stress which are either within or on the yield surface are called "allowable". No state of stress can lie outside the yield surface. The intersection of this prism by a general plane σ_3 = constant is an

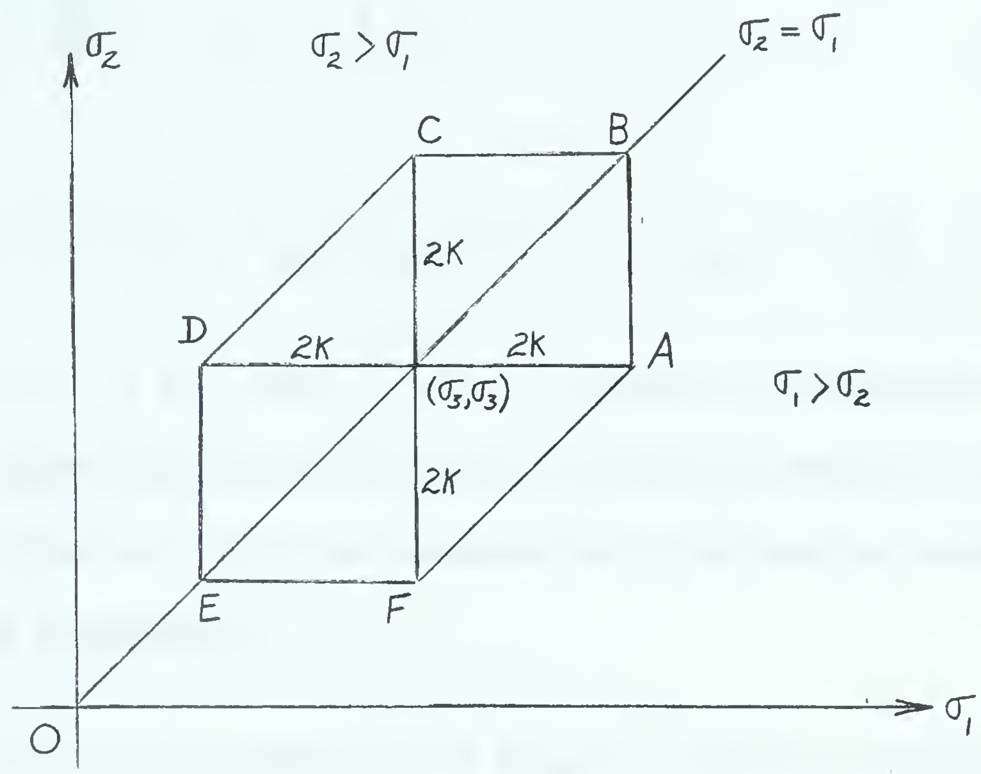


Figure 2
The Tresca Yield Criterion

irregular hexagon. Figure 2 is the orthogonal projection onto the $\sigma_1 - \sigma_2$ plane of such an intersection - a distance σ_3 from the origin 0. Points on this irregular hexagon ABCDEF represent all possible states of stress with maximum shearing stress k. Homogeneity of the material requires that any element of the material will become plastic when the maximum shearing stress on that element has attained the value of this material constant k.

According to the concept of plastic potential (Hill [1950c]), the principal plastic strain-rate free vector $2G\bar{\epsilon}(\epsilon_i)$, in principal stress space, associated with the principal stress bound vector $\bar{\sigma}(\sigma_i)$, has the same direction as the outward normal to the regular yield surface, $f(\sigma_i) = 0$,

at the point $\sigma_i (i = 1,2,3)$ which represents the plastic state of stress. The factor 2G is used only to give dimensions of stress to the plastic strain-rate vector $\bar{\epsilon}(\epsilon_i)$ in the principal stress space. The plastic strain-rate components are determined by

$$\epsilon_i = \lambda \frac{\partial f}{\partial \sigma_i} , \quad (i = 1,2,3) \quad , \tag{2.2.1}$$

where

$\lambda = 0$ if $f < 0$ and also if $f = 0$ and $\dot{f} \equiv \frac{\partial f}{\partial \sigma_i} \dot{\sigma}_i < 0$;

$\lambda \geq 0$ if $f = 0$ and $\dot{f} = 0$; λ being an indeterminate scalar parameter. An upper dot associated with a quantity denotes differentiation with respect to time or any other monotonically increasing parameter correlated with progressive deformation.

Singular yield surfaces with edges or corners are represented by a finite or infinite number of yield surfaces $f_\alpha(\sigma_i)$, $\alpha = 1,2,\ldots$. States of stress at the yield surface are described by a value zero of one or more of the yield functions; all other yield functions being negative. By the Koiter-Prager generalization of the plastic potential (Koiter [1960]), the direction of the plastic strain-rate vector $2G\bar{\epsilon}(\epsilon_i)$ is unique and coincident with the outward normal to the yield surface except at singular points along the edges. At a singular point, the direction of $2G\bar{\epsilon}(\epsilon_i)$ must lie between, and in the plane defined by, the unique normals drawn outwards to the two faces of the prism intersection at the particular singular point considered. The plastic strain-rate components are determined by the formulae

$$\epsilon_i = \lambda_\alpha \frac{\partial f_\alpha}{\partial \sigma_i} , \quad (i = 1,2,3) , \tag{2.2.2}$$

where

$\lambda_\alpha = 0$ if $f_\alpha < 0$ and also if $f_\alpha = 0$, $\dot{f}_\alpha = \dfrac{\partial f_\alpha}{\partial \sigma_i} \dot{\sigma}_i < 0$;

$\lambda_\alpha \geq 0$ if $f_\alpha = 0$ and $\dot{f}_\alpha = 0$; λ_α being indeterminate scalar parameters.

In this thesis, discussion of the types of plastic flow possible as the stress point σ_i traverses the Tresca yield locus is confined to EFAB (Fig. 2) along which $\sigma_1 \geq \sigma_2$. The locus EFAB is divided into 4 distinct groups of plastic regimes; namely: I, B and E; II, AB and EF; III, A and F; and IV, AF. Groups I and III represent singular edge plastic regimes while groups II and IV represent regular face plastic regimes. The yield functions corresponding to the regular plastic regimes are

$$
\begin{aligned}
f_{AB} &= \sigma_1 - \sigma_3 - 2k , \\
f_{AF} &= \sigma_1 - \sigma_2 - 2k , \qquad\qquad (2.2.3) \\
f_{EF} &= \sigma_1 - \sigma_3 + 2k .
\end{aligned}
$$

The plastic strain rates for the 4 groups are obtained by applying (2.2.2) to the yield functions (2.2.3). They are represented in Table I.

Table I

Yield Conditions and Flow Rules for Individual Plastic Regimes

Group	Plastic Regime	Yield Condition	Plastic Strain Rates ϵ_1,	ϵ_2,	ϵ_3
I	B	$\sigma_1 = \sigma_2 = \sigma_3 + 2k$	λ_1,	λ_2,	$-\lambda_1 - \lambda_2$
	E	$\sigma_1 = \sigma_2 = \sigma_3 - 2k$	$-\lambda_1$,	$-\lambda_2$,	$\lambda_1 + \lambda_2$
II	AB	$\sigma_1 = \sigma_3 + 2k$, $\sigma_1 > \sigma_2 > \sigma_3$	λ_1,	0,	$-\lambda_1$
	EF	$\sigma_2 = \sigma_3 - 2k$, $\sigma_3 > \sigma_1 > \sigma_2$	0,	$-\lambda_2$,	λ_2
III	AF	$\sigma_1 = \sigma_2 + 2k$, $\sigma_1 > \sigma_3 > \sigma_2$	λ_3,	$-\lambda_3$,	0
IV	A	$\sigma_1 = \sigma_2 + 2k$, $\sigma_2 = \sigma_3$	$\lambda_1 + \lambda_3$,	$-\lambda_3$,	$-\lambda_1$
	F	$\sigma_1 = \sigma_2 + 2k$, $\sigma_1 = \sigma_3$	λ_3,	$-\lambda_3 - \lambda_2$,	λ_2

The circumferential direction is a principal direction, since $\sigma_{\theta z} = 0$. For purpose of discussion, let $\sigma_3 = \sigma_\theta$ and $\epsilon_3 = \epsilon_\theta$. If either $\sigma_1 \neq \sigma_2$ or $\epsilon_1 \neq \epsilon_2$, then the principal directions of stress and plastic strain-rate within the reference plane are unique and coincident since the material is assumed isotropic. By assumption, $\sigma_1 \geq \sigma_2$ and by isotropy, $\epsilon_1 \geq \epsilon_2$. The positive, first and second, principal directions, σ_1 and σ_2, are then defined to make an angle φ, $0 \leq \varphi < \pi$ with the positive r and z directions respectively. These principal stresses in the $r - z$ reference plane are

$$\sigma_1 = \frac{1}{2}(\sigma_r + \sigma_z) + \{\frac{1}{4}(\sigma_r - \sigma_z)^2 + \sigma_{rz}^2\}^{\frac{1}{2}} \quad ,$$

$$\text{and} \qquad \sigma_2 = \frac{1}{2}(\sigma_2 + \sigma_z) - \{\frac{1}{4}(\sigma_r - \sigma_z)^2 + \sigma_{rz}^2\}^{\frac{1}{2}} \quad . \tag{2.2.5}$$

The principal plastic strain-rate components are

$$\epsilon_1 = \frac{1}{2}(\epsilon_r + \epsilon_z) + \{\frac{1}{4}(\epsilon_r - \epsilon_z)^2 + \epsilon_{rz}^2\}^{\frac{1}{2}} \quad ,$$

$$\epsilon_2 = \frac{1}{2}(\epsilon_r + \epsilon_z) - \{\frac{1}{4}(\epsilon_r - \epsilon_z)^2 + \epsilon_{rz}^2\}^{\frac{1}{2}} \quad , \tag{2.2.6}$$

$$\epsilon_3 = \epsilon_\theta \quad .$$

The stress components in the r and z directions are

$$\sigma_r = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\varphi \quad ,$$

$$\sigma_z = \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\varphi \quad , \tag{2.2.7}$$

$$\sigma_{rz} = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\varphi \quad ,$$

$$\text{where} \qquad \cos 2\varphi = \frac{\frac{1}{2}(\sigma_r - \sigma_z)}{\{\frac{1}{4}(\sigma_r - \sigma_z)^2 + \sigma_{rz}^2\}^{\frac{1}{2}}} \quad , \text{ and}$$

$$\sin 2\varphi = \frac{\sigma_{rz}}{\{\frac{1}{4}(\sigma_r - \sigma_z)^2 + \sigma_{rz}^2\}^{\frac{1}{2}}} \quad .$$

The analogous plastic strain-rate components in the r and z directions are

$$\epsilon_r = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2)\cos 2\varphi \quad ,$$

$$\epsilon_z = \frac{1}{2}(\epsilon_1 + \epsilon_2) - \frac{1}{2}(\epsilon_1 - \epsilon_2)\cos 2\varphi \quad , \tag{2.2.8}$$

$$\epsilon_{rz} = \frac{1}{2}(\epsilon_1 - \epsilon_2)\sin 2\varphi \quad ,$$

where $\cos 2\varphi = \dfrac{\frac{1}{2}(\epsilon_r - \epsilon_z)}{\{\frac{1}{4}(\epsilon_r - \epsilon_z)^2 + \epsilon_{rz}^2\}^{\frac{1}{2}}}$, and

$$\sin 2\varphi = \frac{\epsilon_{rz}}{\{\frac{1}{4}(\epsilon_r - \epsilon_z)^2 + \epsilon_{rz}^2\}^{\frac{1}{2}}} \ .$$

The condition of isotropy is conveniently expressed by the relationships

$$\frac{\sigma_r - \sigma_z}{\epsilon_r - \epsilon_z} = \frac{\sigma_{rz}}{\epsilon_{rz}} = \frac{\sigma_1 - \sigma_2}{\epsilon_1 - \epsilon_2} \tag{2.2.9}$$

which follows from (2.2.7) and (2.2.8). The condition of incompressibility is ~~defined by~~

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon_r + \epsilon_z + \epsilon_\theta = 0. \tag{2.2.10}$$

The general field equations for quasi-static, axially symmetric plastic flow for the type of material considered ~~is~~ are, therefore, given by equations (2.1.1), (2.1.2), and Table I. Equations (2.2.9) and (2.2.10) must also be satisfied by the stress and plastic strain-rate components.

CHAPTER III

ANALYSIS OF THE FIELD EQUATIONS

The field equations of Chapter II are specialized into each of the various plastic regimes of groups I to IV and a mathematical analysis of their structure obtained.

3.1 Group I: Plastic Regimes B and E.

The regimes B and E of Fig. 2 are singular and semi-isotropic since $\sigma_1 = \sigma_2 \neq \sigma_3$. B differs from E only in that it corresponds to a higher mean value of the principal stresses for some given value of the circumferential stress $\sigma_3 = \sigma_\theta$.

(a) Stress Fields.

The yield condition for B (Table I) is $\sigma_1 = \sigma_2 = \sigma_3 + 2k$. Hence, $\sigma_\theta = \sigma_r - 2k$ since by (2.2.7), $\sigma_r = \sigma_z$ and $\sigma_{rz} = 0$. Therefore,

$$\frac{\partial \sigma_r}{\partial r} + \frac{2k}{r} = 0 \quad \text{by (2.1.1) and}$$

$$\begin{aligned} \sigma_r &= \sigma_z = 2k \,\ell n \frac{A}{r}, \quad A > 0 \,, \\ \sigma_\theta &= 2k \,[\,\ell n \,(\tfrac{A}{r}) - 1] \;. \end{aligned} \qquad (3.1.1)$$

This stress field has a singularity at $r = 0$ and, consequently, must be defined only for $r > 0$ to avoid infinities in the stresses.

The stress field for E can be derived in a similar manner; the only component being different is σ_θ, where

$$\sigma_\theta = 2k \,[\,\ell n \,(\tfrac{A}{r}) + 1\,] \;.$$

(b) Velocity Fields.

From Table I, the plastic strain-rate components for B are

$$\epsilon_1 = \lambda_1 \geq 0, \quad \epsilon_2 = \lambda_2 \geq 0 \quad \text{and} \quad \epsilon_3 = -\lambda_1 - \lambda_2 \leq 0 \quad .$$

Since $\epsilon_3 = \epsilon_\theta = \frac{u}{r}$ by (2.1.3), the radial velocity u must be zero or negative for $r > 0$. There being no shearing stress in the $r - z$ plane, the principal axes of stress are not uniquely determined and the isotropy condition (2.2.9) cannot be used as a relationship for determining the principal strain-rates. Consequently, the velocity field is not determinate, being governed only by the single equation of incompressibility (2.2.10). The velocity components u and w must, however, be continuous functions of r and z. This follows as discontinuities in tangential velocities can occur only across a surface on which the shearing stress is k.

The velocity field at E is similarly indeterminate. The component u, however, is either zero or positive as follows from Table I.

3.2 Group II: Plastic Regimes AB and EF.

The plastic regimes AB and EF are regular. Because determination of the velocity fields and stress fields for these regimes is similar, attention is confined only to AB. The outstanding difference between these two regular regimes is in the type of plastic flow possible in the axial planes.

From Table I, the principal strain-rate components for AB are

$$\epsilon_1 = \lambda_1 \geq 0, \quad \epsilon_2 = 0, \quad \text{and} \quad \epsilon_3 = -\lambda_1 \leq 0 \; .$$

Hence, $\epsilon_3 = \epsilon_\theta = \frac{u}{r} \leq 0$ and this implies that $u \leq 0$ for all $r > 0$. The

principal strain-rate components for EF are

$$\epsilon_1 = 0, \quad \epsilon_2 = -\lambda_2 \leq 0, \quad \text{and} \quad \epsilon_3 = \lambda_2 \geq 0 \quad .$$

Hence, $\epsilon_3 = \epsilon_\theta = \frac{u}{r} \geq 0$ and this implies that $u \geq 0$ for all $r > 0$. The regime AB corresponds to states of stress where "waisting" occurs in the axial planes, in contrast to the regime EF where "barrelling" occurs in the axial planes.

(a) Velocity Field for AB.

The condition that $\epsilon_2 = 0$ requires that

$$\epsilon_r + \epsilon_z = \{ (\epsilon_r - \epsilon_z)^2 + 4\epsilon_{rz}^2 \}^{\frac{1}{2}}$$

by (2.2.6). Using (2.1.3), this can be written as

$$\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right)^2 = \left(\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 . \qquad (3.2.1)$$

Also by (2.1.3), the incompressibility equation (2.2.10) can be written as

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 . \qquad (3.2.2)$$

From the yield condition for AB (Table I), $\sigma_1 = \sigma_3 + 2k$, $\sigma_1 > \sigma_2 > \sigma_3$, the further condition that $\frac{\sigma_1 - \sigma_2}{2} < k$ shows that the maximum shearing stress in the $r - z$ plane must be less than k. Consequently, velocity components u and w are to be continuous functions of r and z. These velocity components are determined by (3.2.1) and (3.2.2) with u having the additional requirement of being non-negative for all $r > 0$. The velocity field for AB is kinematically determinate in the sense that there is available as many equations involving the velocity components as there are unknown velocity components.

The equation of incompressibility (3.2.2) can be written

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 . \qquad (3.2.3)$$

If u and w are derivable from a velocity function $V(r,z)$ such that

$$u = \frac{1}{r}\frac{\partial V}{\partial z} \quad \text{and} \quad w = -\frac{1}{r}\frac{\partial V}{\partial r} , \qquad (3.2.4)$$

then, clearly, V satisfies (3.2.3). V must also satisfy (3.2.1). Substitution of (3.2.4) into (3.2.1) gives

$$\frac{1}{r^4}\left(\frac{\partial V}{\partial z}\right)^2 = \left(-\frac{1}{r^2}\frac{\partial V}{\partial z} + \frac{2}{r}\frac{\partial^2 V}{\partial r \partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial^2 V}{\partial z^2} + \frac{1}{r^2}\frac{\partial V}{\partial r} - \frac{1}{r}\frac{\partial^2 V}{\partial r^2}\right)^2 ,$$

which can be simplified to

$$\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)^2 + 4\frac{\partial^2 V}{\partial r \partial z}\left(\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right) = 0 , \qquad (3.2.5)$$

a non-linear second-order partial differential equation in V. If u, w and V are prescribed as initial conditions on some open curve Γ in the $r - z$ plane, then these conditions together with (3.2.5) constitute a general Cauchy problem in the theory of partial differential equations (Petrovsky [1961]). The classification of (3.2.5) and the possible extension of V for points in the immediate neighbourhood of Γ by a Taylor series is now considered.

The following relationships must be satisfied along Γ ; namely,

$$\begin{aligned} d\left(\frac{\partial V}{\partial r}\right) &= \frac{\partial^2 V}{\partial r^2} dr + \frac{\partial^2 V}{\partial r \partial z} dz , \\ d\left(\frac{\partial V}{\partial z}\right) &= \frac{\partial^2 V}{\partial r \partial z} dr + \frac{\partial^2 V}{\partial z^2} dz . \end{aligned} \qquad (3.2.6)$$

Together with (3.2.5) these relationships are sufficient for the determination of the second-order derivatives of V provided they exist. Similarly for the second-order differential coefficients along Γ, there are the relationships

$$d\left(\frac{\partial^2 V}{\partial r^2}\right) = \frac{\partial^3 V}{\partial r^3}\,dr + \frac{\partial^3 V}{\partial r^2 \partial z}\,dz ,$$

$$d\left(\frac{\partial^2 V}{\partial r \partial z}\right) = \frac{\partial^3 V}{\partial r^2 \partial z}\,dr + \frac{\partial^3 V}{\partial r \partial z^2}\,dz , \qquad (3.2.7)$$

$$d\left(\frac{\partial^2 V}{\partial z^2}\right) = \frac{\partial^3 V}{\partial r \partial z^2}\,dr + \frac{\partial^3 V}{\partial z^3}\,dz .$$

In addition, there is the relationship obtained by differentiating (3.2.5) partially with respect to r yielding

$$\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)\left[\frac{\partial^3 V}{\partial r^3} - \frac{\partial^3 V}{\partial z^2 \partial r} + \frac{1}{r^2}\frac{\partial V}{\partial r} - \frac{1}{r}\frac{\partial^2 V}{\partial r^2}\right] \qquad (3.2.8)$$
$$+ 2\frac{\partial^3 V}{\partial r^2 \partial z}\left[\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right] + 2\frac{\partial^2 V}{\partial r \partial z}\left[\frac{\partial^3 V}{\partial r^2 \partial z} + \frac{1}{r^2}\frac{\partial V}{\partial z} - \frac{1}{r}\frac{\partial^2 V}{\partial r \partial z}\right] = 0 ,$$

a quasi-linear partial differential equation of the third-order. Equations (3.2.7) and (3.2.8) are sufficient for the determination of the third-order partial derivatives of V provided the coefficient determinant is non-zero. Curves for which this determinant is zero are the characteristics of the system (3.2.5) and an extension of V is not possible from them as a Taylor series for V cannot be formed (Schiffer [1960]). The characteristics of (3.2.5) are found by the determinantal equation

$$\begin{vmatrix} \frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r} & 2\left(\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right) + 2\frac{\partial^2 V}{\partial r \partial z} & -\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) & 0 \\ dr & dz & 0 & 0 \\ 0 & dr & dz & 0 \\ 0 & 0 & dr & dz \end{vmatrix} = 0 ,$$

which upon expansion becomes

$$\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)dz^2 - 2\left(2\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right)dr\,dz - \left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)dr^2 = 0$$

(3.2.9)

Letting $A(V,r,z) = \frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}$ and $B(V,r,z) = 4\frac{\partial^2 V}{\partial r \partial z} - \frac{2}{r}\frac{\partial V}{\partial z}$, equation (3.2.9) becomes

$$dz^2 - \frac{B}{A}\, dz\, dr - dr^2 = 0 . \tag{3.2.10}$$

Now (3.2.5) can be written as

$$A^2 + 2\frac{\partial^2 V}{\partial r \partial z}\left(B - 2\frac{\partial^2 V}{\partial r \partial z}\right) = 0 .$$

Hence

$$B = \frac{4\left(\frac{\partial^2 V}{\partial r \partial z}\right)^2 - A^2}{2\,\frac{\partial^2 V}{\partial r \partial z}}$$

and

$$\frac{B}{A} = \frac{2\left(\frac{\partial^2 V}{\partial r \partial z}\right)}{A} - \frac{A}{2\left(\frac{\partial^2 V}{\partial r \partial z}\right)} . \tag{3.2.11}$$

Substitution of (3.2.11) into (3.2.10) yields

$$dz^2 - \left[\frac{2\left(\frac{\partial^2 V}{\partial r \partial z}\right)}{A} - \frac{A}{2\left(\frac{\partial^2 V}{\partial r \partial z}\right)}\right] dr\, dz - dr^2 = 0 ,$$

which factors into

$$\left[dz - \frac{2\left(\frac{\partial^2 V}{\partial r \partial z}\right)}{A}\, dr\right]\left[dz + \frac{A}{2\,\frac{\partial^2 V}{\partial r \partial z}}\, dr\right] = 0 .$$

Hence, either

$$\frac{dz}{dr} = \frac{2\,\frac{\partial^2 V}{\partial r \partial z}}{\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}} ,$$

or

$$\frac{dz}{dr} = \frac{\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)}{-2\,\frac{\partial^2 V}{\partial r \partial z}} . \tag{3.2.12}$$

By use of (3.2.5), equation (3.2.12) can be written as

$$\frac{dz}{dr} = \frac{2\left(\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right)}{\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}} .$$

The characteristics are thus defined by the equations

$$\frac{1}{2}\frac{dz}{dr}\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) = \frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z} , \qquad (3.2.13a)$$

and $$\frac{1}{2}\frac{dz}{dr}\left(\frac{\partial^2 V}{\partial r^2} - \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) = \frac{\partial^2 V}{\partial r \partial z} . \qquad (3.2.13b)$$

The partial differential equation (3.2.5) is therefore hyperbolic. Also the two families of characteristics (3.2.13a) and (3.2.13b) form an orthogonal net since the product of their respective slopes is -1.

From Table I, the plastic strain-rates for AB are

$$\epsilon_1 = \lambda_1 \geq 0 , \quad \epsilon_2 = 0 , \quad \epsilon_3 = -\lambda_1 \leq 0 .$$

Hence by (2.2.8),

$$\epsilon_r = \frac{1}{2}\lambda_1 (1 + \cos 2\varphi) ,$$

$$\epsilon_z = \frac{1}{2}\lambda_1 (1 - \cos 2\varphi) ,$$

$$\epsilon_{rz} = \frac{1}{2}\lambda_1 \sin 2\varphi .$$

Therefore $$\frac{\epsilon_{rz}}{\epsilon_r} = \frac{\sin 2\varphi}{1+\cos 2\varphi} = \tan\varphi .$$

Since $$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

$$= \frac{1}{2r}\left(\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial r^2}\right),$$

and $$\epsilon_r = \frac{\partial u}{\partial r}$$

$$= \frac{1}{r}\left(\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}\right) \quad \text{by (3.2.4)},$$

$$\frac{\epsilon_{rz}}{\epsilon_r} = \frac{\frac{1}{2}\left(\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial r^2}\right)}{\frac{\partial^2 V}{\partial r \partial z} - \frac{1}{r}\frac{\partial V}{\partial z}}$$

$$= - \frac{1}{\frac{dz}{dr}} \quad \text{by (3.2.13a)}.$$

Therefore $\frac{dz}{dr} = - \cot \varphi$. (3.2.14)

Similarly, since $\epsilon_z = \frac{\partial w}{\partial z}$

$$= - \frac{1}{r}\frac{\partial^2 V}{\partial r \partial z},$$

$$\frac{\epsilon_{rz}}{\epsilon_z} = \frac{\frac{1}{2r}\left(\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial r^2}\right)}{- \frac{1}{r}\frac{\partial^2 V}{\partial r \partial z}}$$

$$= \frac{1}{\frac{dz}{dr}} \quad \text{by (3.2.13b)}.$$

But $\frac{\epsilon_{rz}}{\epsilon_z} = \frac{\sin 2\varphi}{1 - \cos 2\varphi} = \cot \varphi$.

Therefore $\frac{dz}{dr} = \tan \varphi$. (3.2.15)

From (3.2.14) and (3.2.15), it is seen that the characteristic directions are the same as those of the principal strain-rates in the $r - z$ plane.

(b) Stress Field for AB.

Structure of the stress field for plastic regime AB is determined from the equations of equilibrium (2.1.1) and (2.1.2), the yield condition from Table I, and the isotropy condition (2.2.9).

Consider $\Phi(r,z)$ as a stress function such that

$$\sigma_{rz} = \frac{1}{r}\frac{\partial\Phi}{\partial z} \quad \text{and} \quad \sigma_z = -\frac{1}{r}\frac{\partial\Phi}{\partial r} \quad . \tag{3.2.16}$$

$\Phi(r,z)$ satisfies the equilibrium equation (2.1.2); namely,

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \; .$$

The isotropy condition (2.2.9) can be written

$$\frac{\sigma_r - \sigma_z}{\sigma_{rz}} = \frac{\epsilon_r - \epsilon_z}{\epsilon_{rz}}$$

$$= 2 \cot 2\varphi \quad \text{by} \quad (2.2.8) \; .$$

Therefore

$$\sigma_r - \sigma_z = 2\sigma_{rz} \cot 2\varphi$$

$$\sigma_r = \frac{1}{r}\left\{ (2 \cot 2\varphi) \; \frac{\partial\Phi}{\partial z} - \frac{\partial\Phi}{\partial r} \right\} \quad , \tag{3.2.17}$$

in terms of the stress function $\Phi(r,z)$. The yield condition to be satisfied is $\sigma_1 - \sigma_3 = 2k$ or, equivalently, by use of (2.2.15) that

$$\frac{1}{2}(\sigma_r - \sigma_z) + \{ \frac{1}{4}(\sigma_r - \sigma_z)^2 + \sigma_{rz}^2 \}^{\frac{1}{2}} - \sigma_\theta = 2k \; .$$

Expressed in terms of $\Phi(r,z)$, this yield condition becomes

$$\frac{1}{r}\frac{\partial\Phi}{\partial z} (\cot 2\varphi + \csc 2\varphi) - \sigma_\theta = 2k \; .$$

Hence

$$\sigma_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial z} \cot \varphi - 2k \; . \tag{3.2.18}$$

The equilibrium equation (2.1.1) expressed in terms of $\Phi(r,z)$ now becomes

$$\frac{\partial^2\Phi}{\partial r^2} - \frac{\partial^2\Phi}{\partial z^2} - 2 \cot 2\varphi \frac{\partial^2\Phi}{\partial r \partial z} + \frac{\cot \varphi}{r}\frac{\partial\Phi}{\partial z} - 2k = 0 \; , \tag{3.2.19}$$

a linear second-order partial differential equation in Φ. The characteristic directions are determined using (3.2.19) and equations (3.2.20); namely,

$$\begin{aligned} d\left(\frac{\partial\Phi}{\partial r}\right) &= \frac{\partial^2\Phi}{\partial r^2}\,dr + \frac{\partial^2\Phi}{\partial r\partial z}\,dz\,, \\ d\left(\frac{\partial\Phi}{\partial z}\right) &= \frac{\partial^2\Phi}{\partial r\partial z}\,dr + \frac{\partial^2 V}{\partial z^2}\,dz\,, \end{aligned} \qquad (3.2.20)$$

which expresses the variation of the first derivatives of Φ along any curve Γ on the r,z plane. Equating to zero the coefficient determinant of (3.2.19) and (3.2.20); i.e.

$$\begin{vmatrix} 1 & -2\cot 2\varphi & -1 \\ dr & dz & 0 \\ 0 & dr & dz \end{vmatrix} = 0\,,$$

yields upon expansion

$$dz^2 + 2\cot 2\varphi\, dr\, dz - dr^2 = 0\,.$$

Thus $\quad \dfrac{dz}{dr} = \tan\varphi,$

and $\quad \dfrac{dz}{dr} = -\cot\varphi$ are the characteristic directions.

The above analysis shows that the partial differential equation (3.2.19) is hyperbolic and that the characteristics form an orthogonal net coincident with the characteristics of the partial differential equation (3.2.5) for the velocity function $V(r,z)$.

In conclusion, therefore, the plastic regime AB (similarly for EF) has a velocity field which is kinematically determinate and hyperbolic and a stress field which is statically determinate and hyperbolic. The characteristics of both fields are coincident and have the same directions as the principal strain-rates in the r,z plane.

3.3 Group III: Plastic Regimes A and F.

The plastic regimes A and F in Figure 2 are singular and characterized by the equality of the circumferential principal stress with one of the two principal stresses in the meridian plane. This is the "full plasticity (vollplastisch)" hypothesis proposed by Haar and von Kármán in 1909.

(a) Stress Field for F.

Consider regime F for definiteness. There are two equilibrium equations (2.1.1) and (2.1.2) and two yield conditions from Table I available for the determination of the four stress components σ_r, σ_θ, σ_z and σ_{rz}. In this sense, the stress field is statically determinate.

From the yield conditions $\sigma_1 - \sigma_2 = 2k$ and $\sigma_3 - \sigma_2 = 2k$ where $\sigma_3 = \sigma_\theta$, it follows that

$$\left\{ \frac{1}{4} (\sigma_r - \sigma_z)^2 + \sigma_{rz}^2 \right\}^{\frac{1}{2}} = k \quad \text{by the use of (2.2.5).}$$

Hence the maximum shearing stress in the meridian plane is k. The yield conditions impose two independent conditions on the four stress components and, consequently, the possible yield states may be represented by two independent parameters. These parameters are taken as the pressure

$$\begin{aligned} p(r,z) &= -\frac{1}{2} (\sigma_1 + \sigma_2) \\ &= -\frac{1}{2} (\sigma_r + \sigma_z) \end{aligned} \tag{3.3.1}$$

and the angle $\varphi(r,z)$ which specifies the orientation of the principal axes in the meridian plane. The lines of maximum shearing stress (slip lines) form an orthogonal system of curvilinear coordinates at a point in the meridian plane. By convention, these lines are called α- and β-lines and the direction of the algebraically greater principal stress σ_1 is

obtained by an anticlockwise rotation of an angle $\pi/4$ from the α - line. The inclination of the α - line at any point with the r-axis is $\varphi(r,z)$. Since surface elements perpendicular to the α, β - lines are acted upon by a shearing stress k and a normal stress $-p$,

$$\sigma_\alpha = -p, \quad \sigma_\beta = -p, \quad \sigma_{\alpha\beta} = k$$

and

$$\begin{aligned} \sigma_r &= -p - k \sin 2\varphi, \\ \sigma_z &= -p + k \sin 2\varphi, \\ \sigma_{rz} &= k \cos 2\varphi, \\ \sigma_\theta &= -p + k . \end{aligned} \tag{3.3.2}$$

Substitution of (3.3.2) into the equations of equilibrium (2.1.1) and (2.1.2) results in

$$\begin{aligned} &\frac{\partial p}{\partial r} + 2k \cos 2\varphi \frac{\partial \varphi}{\partial r} + 2k \sin 2\varphi \frac{\partial \varphi}{\partial z} = -\frac{k}{r}(1 + \sin 2\varphi) \\ \text{and} \quad &\frac{\partial p}{\partial z} + 2k \sin 2\varphi \frac{\partial \varphi}{\partial r} - 2k \cos 2\varphi \frac{\partial \varphi}{\partial z} = \frac{k}{r} \cos 2\varphi . \end{aligned} \tag{3.3.3}$$

The characteristics of this system of quasi-linear simultaneous equations are determined using (3.3.3) together with the relations

$$\begin{aligned} &dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz , \\ \text{and} \quad &d\varphi = \frac{\partial \varphi}{\partial r} dr + \frac{\partial \varphi}{\partial z} dz \end{aligned} \tag{3.3.4}$$

by equating to zero the coefficient determinant of Cramer's rule for the solution of $\frac{\partial p}{\partial r}$, $\frac{\partial p}{\partial z}$, $\frac{\partial \varphi}{\partial r}$ and $\frac{\partial \varphi}{\partial z}$ (Hildebrand [1954]). Specifically,

$$\begin{vmatrix} 1 & 0 & 2k\cos 2\varphi & 2k\sin 2\varphi \\ 0 & 1 & 2k\sin 2\varphi & -2k\cos 2\varphi \\ dr & dz & 0 & 0 \\ 0 & 0 & dr & dz \end{vmatrix} = 0$$

yields upon expansion the equation

$$(\sin 2\varphi)\, dr^2 - 2\cos 2\varphi\, dr\, dz - (\sin 2\varphi)\, dz^2 = 0 \text{ .}$$

Therefore

$$\frac{dz}{dr} = \tan\varphi$$

$$\text{or} \qquad \frac{dz}{dr} = -\cot\varphi \text{ .} \tag{3.3.5}$$

Accordingly, the characteristics (3.3.5) are real and distinct and the stress equations (3.3.3) are hyperbolic. The characteristics are orthogonal and, moreover, coincide with the lines of maximum shearing stress (α and β - lines).

Requirements to be satisfied along the characteristics (3.3.5) to insure solutions for $\frac{\partial p}{\partial r}$, $\frac{\partial p}{\partial z}$, $\frac{\partial \varphi}{\partial r}$ and $\frac{\partial \varphi}{\partial z}$ are obtained by equating to zero the numerator determinant of Cramer's rule. Specifically, this determinantal equation

$$\begin{vmatrix} 1 & 0 & 2k\cos 2\varphi & -\frac{k}{r}(1+\sin 2\varphi) \\ 0 & 1 & 2k\sin 2\varphi & \frac{k}{r}(\cos 2\varphi) \\ dr & dz & 0 & dp \\ 0 & 0 & dr & d\varphi \end{vmatrix} = 0$$

yields upon expansion

$$dp + \frac{dz}{dr}\left(2k \sin 2\varphi \, d\varphi - \frac{k}{r}\cos 2\varphi \, dr\right) + 2k \cos 2\varphi \, d\varphi + \frac{k}{r}(1 + \sin 2\varphi)\, dr = 0 \; . \qquad (3.3.6)$$

But for an α - line,

$$\frac{dz}{dr} = \tan \varphi, \quad dr = \cos \varphi \, ds_\alpha \quad \text{and} \quad dz = \sin \varphi \, ds_\alpha \; , \qquad (3.3.7)$$

where ds_α is the differential arc-length. Thus for an α - line, there is the requirement that

$$dp + 2k \, d\varphi + k(\sin \varphi + \cos \varphi)\,\frac{ds_\alpha}{r} = 0 \; . \qquad (3.3.8)$$

Similarly for a β - line,

$$\frac{dz}{dr} = -\cot \varphi, \quad dr = -\sin \varphi \, ds_\beta \quad \text{and} \quad dz = \cos \varphi \, ds_\beta \; , \qquad (3.3.9)$$

where ds_β is the differential arc-length. Accordingly for a β - line, there is the requirement that

$$dp - 2k \, d\varphi - k\,(\cos\varphi + \sin \varphi)\,\frac{ds_\beta}{r} = 0 \; . \qquad (3.3.10)$$

(b) Velocity Field for F.

For plastic regime F, the velocity components u and w are determined from the isotropy condition (2.2.9) and the incompressibility condition (2.2.10) or (3.2.2) as it is more conveniently written for purposes of this analysis. The isotropy condition to be satisfied is

$$\frac{\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}}{\dfrac{\partial u}{\partial r} - \dfrac{\partial w}{\partial z}} = -\cot 2\varphi \qquad (3.3.11)$$

as follows from (3.3.2) and (2.2.9). There are, therefore, two equations (3.2.2) and (3.3.11) available for the determination of u and w; namely,

$$\cot 2\varphi \frac{\partial u}{\partial r} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} - \cot 2\varphi \frac{\partial w}{\partial z} = 0$$

and
$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \ . \qquad (3.3.12)$$

Also, there exists the restrictions from Table I that $\epsilon_2 = -\lambda_3 - \lambda_2 \leq 0$ and $\epsilon_3 = \lambda_2 \geq 0$. Since $\epsilon_3 = \epsilon_\theta = \frac{u}{r}$, $u \geq 0$ for $r > 0$. Also $\epsilon_2 \leq 0$ requires that

$$\left\{ \left(\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 \right\}^{\frac{1}{2}} \geq -\frac{u}{r}$$

as follows from (2.1.3), (2.2.6) and (3.2.2).

The characteristics of the velocity field are determined from (3.3.12) and the additional relationships that

$$du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial z} dz$$

and
$$dw = \frac{\partial w}{\partial r} dr + \frac{\partial w}{\partial z} dz \qquad (3.3.13)$$

for any open curve Γ on the r,z plane along which u and w are known. These characteristics are defined by the determinantal equation

$$\begin{vmatrix} \cot 2\varphi & 1 & 1 & -\cot 2\varphi \\ 1 & 0 & 0 & 1 \\ dr & dz & 0 & 0 \\ 0 & 0 & dr & dz \end{vmatrix} = 0 ,$$

which upon expansion yields

$$dz^2 + 2 \cot 2\varphi \, dr \, dz - dr^2 = 0 \ .$$

Therefore $\frac{dz}{dr} = \tan \varphi$

and $\frac{dz}{dr} = -\cot \varphi \ . \qquad (3.3.14)$

Hence the characteristics of the velocity field are orthogonal and coincide with the characteristics of the stress field.

The relationships along these characteristics are obtained from the determinantal equation

$$\begin{vmatrix} 0 & 1 & 1 & -\cot 2\varphi \\ -\frac{u}{r} & 0 & 0 & 1 \\ du & dz & 0 & 0 \\ dw & 0 & dr & dz \end{vmatrix} = 0 .$$

This reduces to

$$\frac{u}{r}\, dz^2 + dw\, dz + du\, dr + \frac{u}{r} \cot 2\varphi\, dz\, dr = 0 . \qquad (3.3.15)$$

By use of (3.3.7), the relationship along an α - characteristic is

$$\cos \varphi\, du + \sin \varphi\, dw + \frac{u}{r} \frac{ds_\alpha}{z} = 0 . \qquad (3.3.16)$$

Similarly by (3.3.9), the relationship along a β - characteristic is

$$\sin \varphi\, du - \cos \varphi\, dw - \frac{u}{r} \frac{ds_\beta}{z} = 0 . \qquad (3.3.17)$$

These relationships (3.3.16) and (3.3.17) are conveniently written in terms of the velocity resolutes U and W. If U is the velocity component along an α - characteristic and W is the velocity component along a β - characteristic, then

$$U = u \cos \varphi + w \sin \varphi ,$$

$$\qquad (3.3.18)$$

and $$W = - u \sin \varphi + w \cos \varphi .$$

From (3.3.18), it follows that

$$dU = \cos\varphi\, du + \sin\varphi\, dw + W d\varphi$$

$$\text{and} \qquad dW = -\sin\varphi\, du + \cos\varphi\, dw - U d\varphi\,. \tag{3.3.19}$$

The characteristic relations (3.3.16) and (3.3.17) now reduce to

$$dU - W d\varphi + \frac{u}{2r}\, ds_\alpha = 0 \quad \text{on an } \alpha - \text{line}, \tag{3.3.20}$$

and

$$dW + U d\varphi + \frac{u}{2r}\, ds_\beta = 0 \quad \text{on an } \beta - \text{line}. \tag{3.3.21}$$

In conclusion, the plastic regime F is an application of the Haar and von Kármán hypothesis. The stress field is statically determinate and hyperbolic with characteristics coincident with the lines of maximum shearing stress. The velocity field is kinematically determinate and likewise hyperbolic with characteristics identical with those of the stress field. These statements apply to the plastic regime A also.

3.4 Group IV: Plastic Regime AF.

(a) Velocity Field for AF.

The plastic strain-rate components for this regular regime are

$$\epsilon_1 = \lambda_3 \geq 0\,, \quad \epsilon_2 = -\lambda_3 \leq 0 \quad \text{and} \quad \epsilon_3 = 0$$

as found in Table I. Since $\epsilon_3 = \frac{u}{r} = 0$, there is no displacement or velocity component u in the radial direction. Moreover, since $\epsilon_r = \frac{\partial u}{\partial r}$, $\epsilon_r = 0$. From the incompressibility equation (3.3.2), it follows that $\epsilon_z = \frac{\partial w}{\partial z} = 0$. Hence $\epsilon_{rz} = \frac{1}{2}\frac{\partial w}{\partial r}$ is the only non-vanishing plastic strain-rate component and for this w must be a function of r only. With ϵ_{rz} being the only non-zero component it means that the r and z directions are the directions of the maximum shearing strain-rate.

(b) Stress Field for AF.

For the determination of the stresses, there is available the yield criterion $\sigma_1 = \sigma_2 + 2k$ where $\sigma_1 > \sigma_3 > \sigma_2$, the isotropy condition (2.2.9) and the equations of equilibrium (2.1.1) and (2.1.2). The yield condition implies that $\frac{\sigma_1 - \sigma_2}{2} = k$. Thus the maximum shearing stress in the meridian plane is k. The isotropy condition (2.2.9) becomes

$$\frac{\pm k}{\epsilon_{rz}} = \frac{\sigma_{rz}}{\epsilon_{rz}} \tag{3.4.1}$$

and $$\sigma_r = \sigma_z \quad . \tag{3.4.2}$$

From (3.4.1), choose $\sigma_{rz} = + k$ for definiteness. Equation of equilibrium (2.1.2) becomes

$$\frac{\partial \sigma_z}{\partial z} + \frac{k}{r} = 0 \; . \tag{3.4.3}$$

Hence $$\sigma_z = \sigma_r = - \frac{k}{r} z + f(r) \; , \tag{3.4.4}$$

where $f(r)$ is an arbitrary function of r. By substitution of σ_z into (2.1.1), there arises the equation

$$\frac{k}{r^2} z + f'(r) - \frac{k}{r^2} z + \frac{f(r)}{r} - \frac{\sigma_\theta}{r} = 0 \; .$$

Hence $$\sigma_\theta = r f'(r) + f(r)$$

or $$\sigma_\theta = \frac{d}{dr} [r f(r)] \; . \tag{3.4.5}$$

The function $f(r)$ must be such that σ_θ is an intermediate principal stress. From (2.2.5), this requires that

$$\frac{1}{2} (\sigma_r + \sigma_z) + k > \sigma_\theta > \frac{1}{2} (\sigma_r + \sigma_z) - k$$

or, equivalently, that

$$-\frac{k}{r}z + k > r\frac{d}{dr}f(r) > -\frac{k}{r}z - k \quad . \qquad (3.4.6)$$

Thus for plastic regime AF_5 the directions of the principal strain-rates and the principal stresses are fixed. The stress field is statically determinate and the velocity field is kinematically determinate. The fact that the radial velocity component is zero renders the application of this regime to limited and rare cases.

CHAPTER IV

INCIPIENT PLASTIC FLOW IN A SEMI-INFINITE REGION OF RIGID-PLASTIC NON-HARDENING ISOTROPIC MATERIAL DUE TO A LOAD APPLIED BY A SMOOTH CONICAL INDENTER FITTED INTO A CONICAL CAVITY

The problem of finding an equilibrium plastic stress field under axially symmetric conditions for indentation by a cone is considered. The yield point load, which is the load at which the material first deforms, is derived using the incomplete stress field. No bounds are made on the actual yield point load since the velocity field and the extended stress field into the rigid region are not determined.

4.1 Preliminary.

The adopted procedure used in this problem is similar to that used by Berezencov [1955], Shield [1955], and Cox, Eason and Hopkins [1961]. Each investigator assumed a priori a plastic regime applicable to their respective problem. Except for Berezencov, they verified that the derived plastic stress field was the correct one by determining a compatible kinematically admissible velocity field. The validity of this procedure has been established by Hill [1951] and Bishop [1953].

Hill [1951] showed by virtual work and the maximum work principle for an element that wherever deformation is occurring in a rigid-plastic non-hardening material, the state of stress is unique in the deforming regions of solutions. In the rigid zone common to all the solutions, the stress need not be unique. Similar conclusions could not be drawn for the velocity field, in fact, it need not be unique anywhere. The velocity field must only be compatible with the stress field. Hill [1950c] has also shown that a correct velocity field is obtained in the plastic zone if upon examination of the stress increment occurring during an increment of

distortion, the elements assumed deformed remain plastically stressed. Actual velocity fields occurring in real metals depend upon factors such as elastic effects and work-hardening. These factors are neglected in calculations for the yield load on rigid-plastic, non-hardening metals.

Bishop [1953] discussed a working procedure for the solutions of two-dimensional kinematically determinate plastic problems. The inverse of Bishop's procedure, as used by Shield [1955], is more applicable to problems that are statically determinate. Under this scheme, one proceeds by ignoring the development of the plastic zone and derives the unique plastic stress field and an associated distortion mode that satisfies the boundary conditions at yield only. This forms an "incomplete" solution in Bishop's terminology. For a "complete" solution a stress field is required in the rigid zone. The stress distributions must satisfy the equilibrium equations and must not violate the yield condition. According to Hill [1951], if the yield condition is locally exceeded, such a solution is applicable for a metal which is at least hardened there by that amount.

Extremum principles established by Markov [1947], Hill [1950d, 1951], Prager [1954] and limit analysis theorems (Drucker, Greenberger and Prager [1952]) can be used on rigid-plastic, non-hardening materials to find bounds for the yield point load. Hill [1951] has shown that any distribution of stress satisfying the equilibrium equations, the stress boundary conditions and nowhere violating the yield criterion gives rise to external boundary loads which are not greater than the actual yield point load. The partial stress field obtained in an incomplete solution does not meet the requirements of this lower bound theorem, consequently, cannot be used to establish a lower bound. However, if it can be demonstrated that an equilibrium stress field (not necessarily unique) that satisfies the

boundary conditions and does not exceed the yield point exists in the rigid region, then this complete stress field can be used to establish a lower bound. The extremum principles also show that the internal rate of plastic work calculated from any velocity field or mode of deformation compatible with any plastic stress field and velocity boundary condition is not less than the rate of working of the actual boundary stresses. Any postulated velocity field, therefore, gives an upper bound to the actual yield point load.

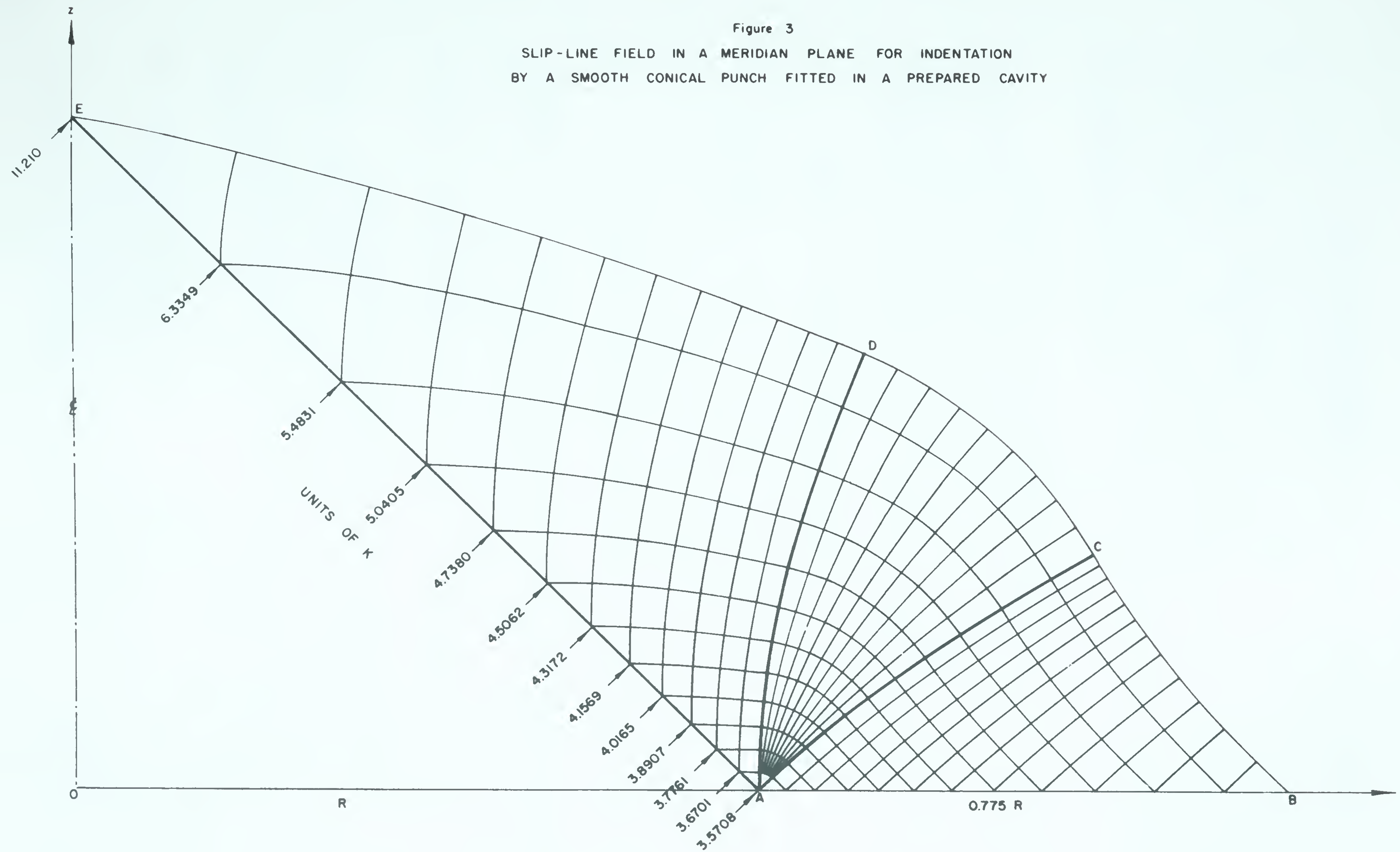
Shield [1955] used the above procedure to solve the Ishlinskii problem referred to in Chapter I. By assuming that the state of stress at yield is represented by the Haar-Kármán plastic regime F of Group III, Chapter III, Shield derived a complete solution and verified it as the actual one. This was attained by showing that the upper and lower bounds for the yield point load were equal.

4.2 Formulation of the Problem.

With reference to Figure 3, for a cylindrical polar coordinate system (r, θ, z) with the z-axes choosen as vertical, let the region $z \geq 0$ be occupied by a rigid-plastic, non-hardening, isotropic material except for the region where a right circular, conical cavity with 45° semi-angle exists such that its base of radius R is on the surface $z = 0$. The axis of the cavity is perpendicular to the surface of the material and the center of the base is choosen as the origin of the coordinate system. A smooth, rigid, right circular, conical punch is inserted into the cavity fitting it fully. The punch is then centrally loaded until incipient plastic flow of the material occurs. The arrangement is clearly one of axial symmetry.

Figure 3

SLIP-LINE FIELD IN A MERIDIAN PLANE FOR INDENTATION BY A SMOOTH CONICAL PUNCH FITTED IN A PREPARED CAVITY



For axial symmetry [Chapter II (2.1)], the circumferential stress σ_θ must be a principal stress. The surface of the material for which $r > R$ is stress free. Consequently $\sigma_z = \sigma_{rz} = 0$ on this surface. This condition requires that σ_r and σ_z be principal stresses for surface elements and that in any meridian plane the lines of maximum shearing stress meet the free boundary at 45° angles.

The state of stress at incipient plastic flow is assumed to be represented by the Haar-Kármán plastic regime F of Group III (Chapter III). Consequently, the equations governing the stress state in the plastic region are hyperbolic and the α - and β - characteristics are given by (3.3.5):

$$\frac{dz}{dr} = \begin{cases} \tan\varphi & \text{on an } \alpha - \text{line} , \\ -\cot\varphi & \text{on a } \beta - \text{line} . \end{cases}$$

The relationships along the characteristics are given by (3.3.8) and (3.3.10):

$$dp + 2k\, d\varphi + k(\sin\varphi + \cos\varphi)\frac{ds_\alpha}{r} = 0 \quad \text{on an } \alpha - \text{line} ,$$

and

$$dp - 2k\, d\varphi - k(\sin\varphi + \cos\varphi)\frac{ds_\beta}{r} = 0 \quad \text{on a } \beta - \text{line} .$$

In Chapter III, the characteristics of this plastic regime were shown to coincide with the lines of maximum shearing stress. Hence they must meet the stress free boundary $z = 0$, $r > R$ at 45° angles.

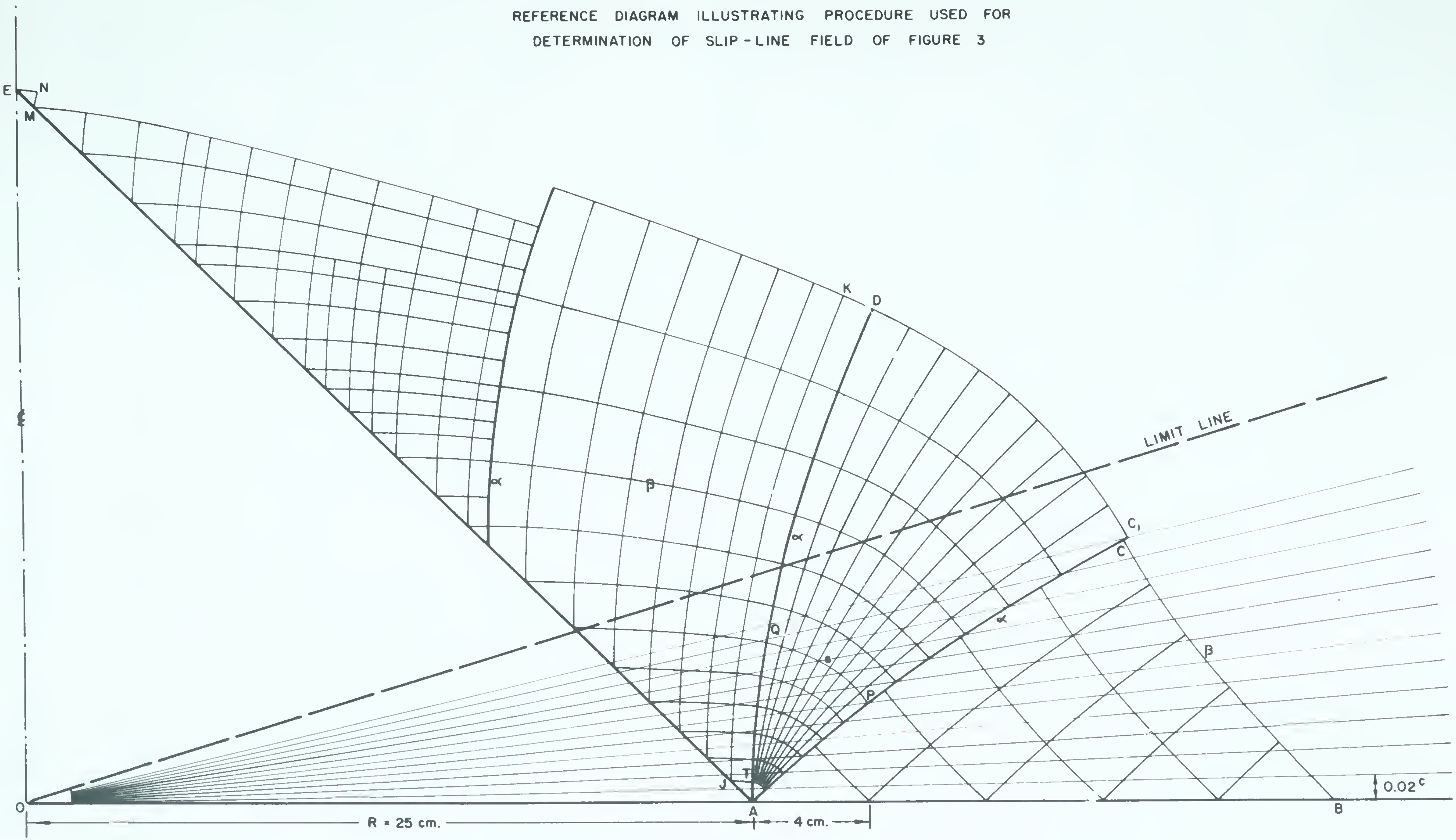
Since the development of the plastic regime is ignored, let the region ABE of Fig. 3 represent one-half of the plastic stress region in a meridional plane at yield; the remaining half of the field follows by symmetry.

4.3 Plastic Stress Field ABC.

From hyperbolic characteristic theory [Schiffer, 1960], the state of stress can be determined in the region ABC (Fig. 4) from the initial

Figure 4

REFERENCE DIAGRAM ILLUSTRATING PROCEDURE USED FOR DETERMINATION OF SLIP-LINE FIELD OF FIGURE 3



conditions on the non-characteristic line AB. Line AC is an α - characteristic through singular end point A and BC is a β - characteristic through end-point B. The position of B (Fig. 3) on the stress-free boundary is not known until it can be determined where B must be such that an extended β - line through B will pass through the E, the apex of the conical cavity. The solution of the stress field in ABC is a Cauchy problem in partial differential equation theory and corresponds to the second boundary-value problem [Hill, 1950e] in plane strain plasticity.

Shield [1955] has derived general equations for the characteristic fields generated by straight-line, stress-free boundaries which are inclined at an angle γ to the r - axis. In particular, plastic stress fields were obtained for $\gamma = 0^{\circ}$, 30° and 60°. The field for $\gamma = 0^{\circ}$ is immediately applicable to the present problem and an analysis of Shield's work is considered here for completeness.

A straight-line boundary in the r - z plane of an axially symmetric body arises when part of the surface of the body is a portion of a circular cone. Figure 5 illustrates the arrangement whereby a cylindrical polar coordinate system is choosen such that its origin coincides with the virtual apex of the conical free surface. The material bordering a straight-line, stress-free boundary is assumed to be in a plastic state of stress represented by point F of Fig. 2 in principal stress space.

$$\text{Let} \quad \psi = \tan^{-1} \frac{z}{r} \quad , \quad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} , \tag{4.3.1}$$

and the free boundary be $\psi = \gamma$, γ a constant. Slip lines meet the free boundary at 45° angles.

The field generated by the free boundary is determined on the assumption that p and φ are functions of ψ only; an assumption

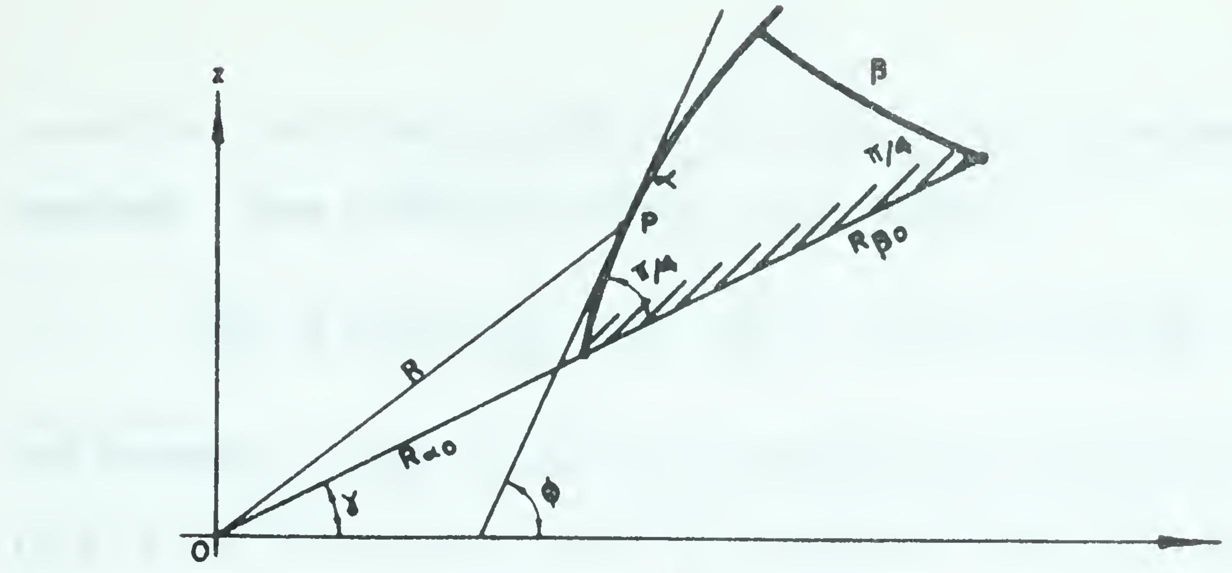


FIGURE 5

STRAIGHT - LINE FREE BOUNDARY

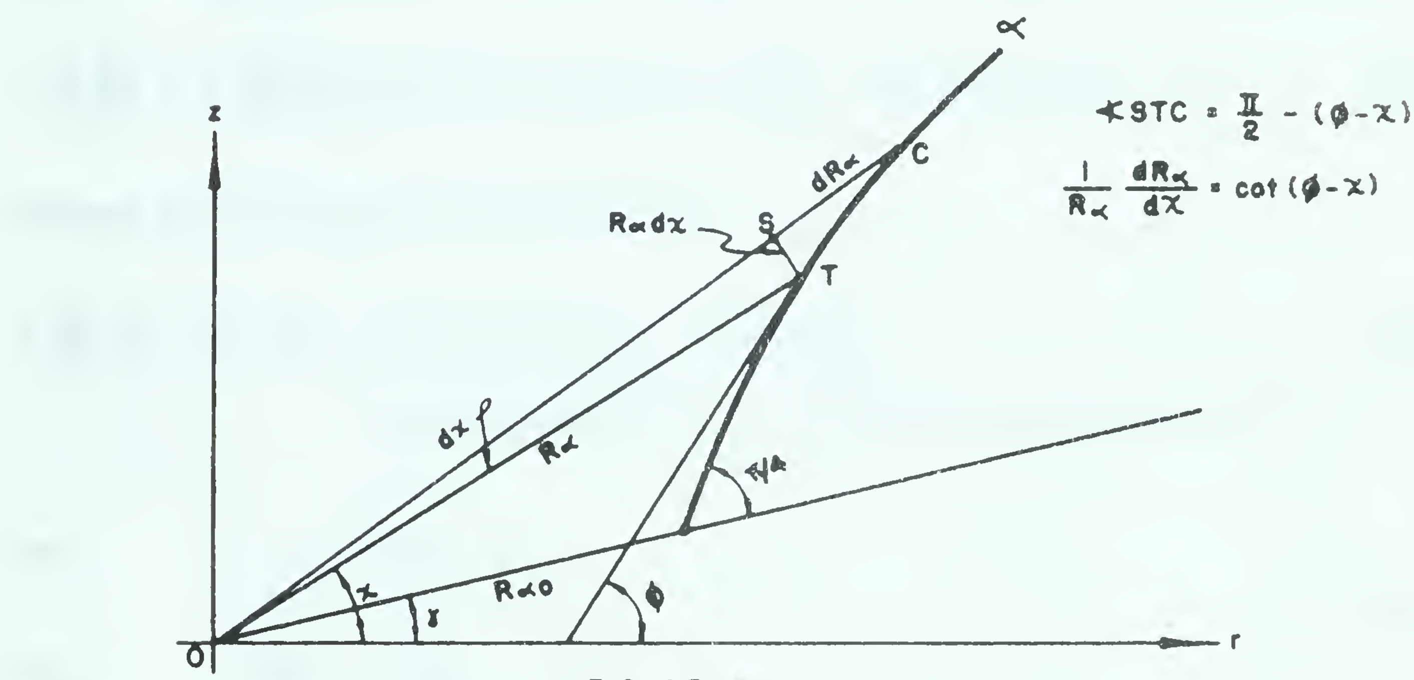


FIGURE 5a.

DIAGRAM FOR DETERMINING $\frac{dR_\alpha}{d\chi}$

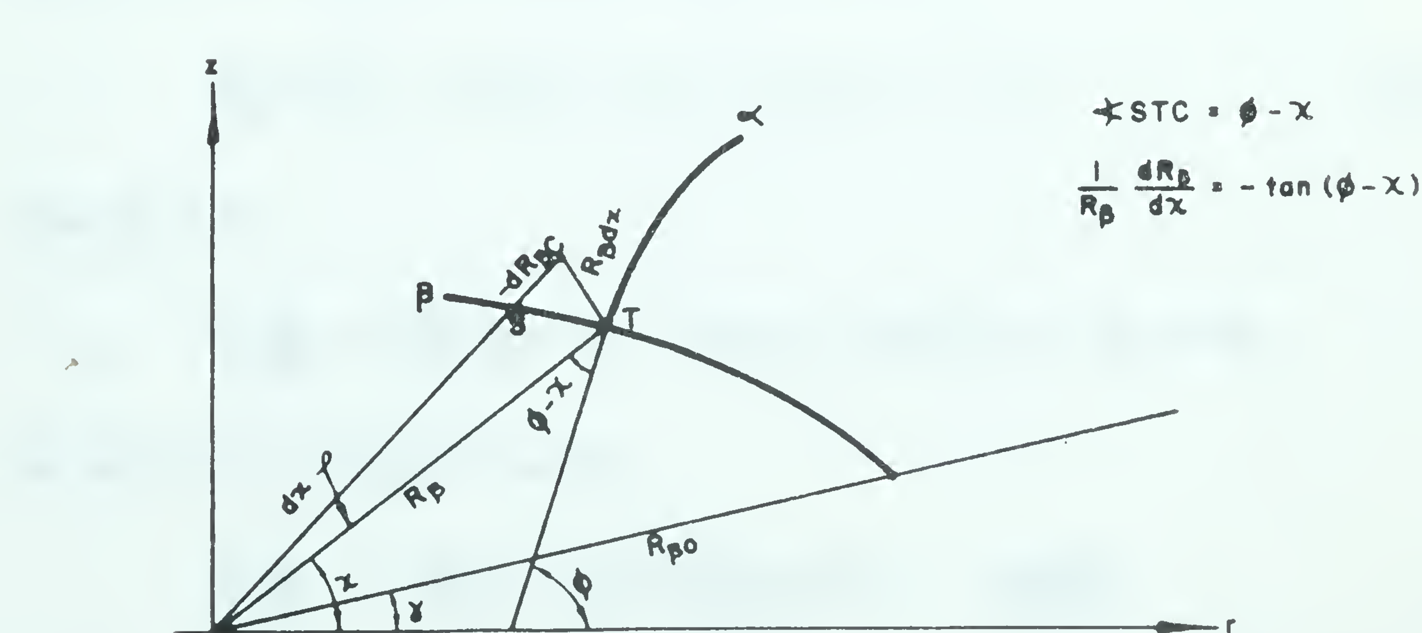


FIGURE 5b.

DIAGRAM FOR DETERMINING $\frac{dR_\beta}{d\chi}$

compatible with the equilibrium equations since no fundamental length is involved. From condition (4.3.1), the operators

$$\frac{\partial}{\partial z} = \frac{1}{r} \cos^2 \psi \frac{d}{d\psi} \quad \text{and} \quad \frac{\partial}{\partial r} = - \frac{1}{r} \sin \psi \cos \psi \frac{d}{d\psi} \tag{4.3.2}$$

are derivable. When (3.3.2) are substituted into equilibrium equations (2.1.1) and (2.1.2) and by use of the operators (4.3.2), there results the differential equations;

$$\frac{1}{k}\frac{dp}{d\psi} + 2\frac{d\varphi}{d\psi}\left\{\cos 2\varphi - \cot\psi \sin 2\varphi\right\} - \left\{1 + \sin 2\varphi\right\} \csc\psi \sec\psi = 0, \tag{4.3.3}$$

$$- \frac{1}{k}\frac{dp}{d\psi} + 2\frac{d\varphi}{d\psi}\left\{\cos 2\varphi + \tan\psi \sin 2\varphi\right\} + \cos 2\varphi \sec^2\psi = 0. \tag{4.3.4}$$

Adding (4.3.3) and (4.3.4) yields

$$\begin{aligned} 2\frac{d\varphi}{d\psi}\left\{2\cos 2\varphi + \sin 2\varphi\,(\tan\psi - \cot\psi)\right\} & \\ + \cos 2\varphi \sec^2\psi - (1 + \sin 2\varphi) \sec\psi \operatorname{cosec}\psi &= 0. \end{aligned} \tag{4.3.5}$$

$$\begin{aligned} \text{Let} \qquad & \Gamma = 2(\varphi - \psi), \\ \text{then} \qquad & 2\frac{d\varphi}{d\psi} = \frac{d\Gamma}{d\psi} + 2. \end{aligned} \tag{4.3.6}$$

Substitution of (4.3.6) into (4.3.5) yields

$$\frac{d\Gamma}{d\psi} \sin\Gamma + 3\sin\Gamma + \cos\Gamma \tan\psi + 1 = 0 . \tag{4.3.7}$$

From (4.3.4),

$$\frac{1}{k}\frac{dp}{d\psi} = 2\frac{d\varphi}{d\psi}\left\{\cos 2\varphi + \tan\psi \sin 2\varphi\right\} + \cos 2\varphi \sec^2\psi .$$

By use of (4.3.6), this becomes

$$\begin{aligned} \frac{1}{k}\frac{dp}{d\psi} &= \left(\frac{d\Gamma}{d\psi} + 2\right)\left\{\frac{\cos(\Gamma + \psi)}{\cos\psi}\right\} + \frac{\cos 2\varphi}{\cos^2\psi} \\ &= \left(\frac{d\Gamma}{d\psi} + 3\right) \frac{\cos(\Gamma + \psi)}{\cos\psi} - \frac{\sin(\Gamma + \psi)\sin\psi}{\cos^2\psi} \end{aligned}$$

$$= \frac{d\Gamma}{d\psi}\cos\Gamma + \cos\Gamma - \sin\Gamma\tan\psi\left(\frac{d\Gamma}{d\psi} + 4\right) - \cos\Gamma\tan^2\psi .$$

But $\quad \dfrac{d\Gamma}{d\psi} + 4 = 1 - \dfrac{1}{\sin\Gamma} - \dfrac{\cos\Gamma\tan\psi}{\sin\Gamma}$ $\quad$ by (4.3.7).

Therefore

$$\frac{1}{k}\frac{dp}{d\psi} = \frac{d\Gamma}{d\psi}\cos\Gamma + 3\cos\Gamma - \sin\Gamma\tan\psi + \tan\psi .$$

Hence

$$\frac{p(\psi)}{k} = \int_{\gamma}^{\psi} (3\cos\Gamma - \sin\Gamma\tan\psi)\, d\psi + \sin\Gamma - \ln\cos\psi + A \tag{4.3.8}$$

where A is determined from the condition that $p = k$ when $\psi = \gamma$.

To plot the α - characteristics, let R_α be the distance from the origin of a point on an α - line and $R_{\alpha o}$ the value of R_α on the boundary $\psi = \gamma$. With reference to Figure 5a, there is derived the relationship

$$\frac{dR_\alpha}{R_\alpha d\psi} = \tan\left[\frac{\pi}{2} - (\varphi - \psi)\right]$$

$$= \cot(\varphi - \psi).$$

Therefore $\quad \ln\dfrac{R_\alpha}{R_{\alpha o}} = \displaystyle\int_{\gamma}^{\psi} \cot(\varphi - \psi)\, d\psi$

or

$$\frac{R_\alpha}{R_{\alpha o}} = \exp\left\{\int_{\gamma}^{\psi} \cot\frac{\Gamma}{2}\, d\psi\right\}. \tag{4.3.9}$$

Similarly, with reference to Figure 5b, if R_β is the distance from the origin of a point on a β - line and $R_{\beta o}$ is the value of R_β on the boundary $\psi = \gamma$, then

$$\frac{dR_\beta}{R_\beta d\psi} = \tan\left[-(\varphi - \psi)\right] .$$

Therefore

$$\frac{R_\beta}{R_{\beta o}} = \exp\left\{-\int_{\gamma}^{\psi} \tan\frac{\Gamma}{2}\, d\psi\right\}. \tag{4.3.10}$$

For the present problem in conical indentation, $\gamma = 0$ and the boundary conditions on $\psi = \gamma = 0$ are $p(\gamma) = k$ and $\varphi = \pi/4$. Using numerical integration, Shield determined $\Gamma(\psi)$ from (4.3.7) together with the initial condition $\Gamma(\gamma) = \pi/2$. Similarly from (4.3.8), $p(\psi)$ was obtained with the initial condition that $p(\gamma) = k$. $\frac{R_\alpha}{R_{\alpha o}}$ and $\frac{R_\beta}{R_{\beta o}}$ were then determined from (4.3.9) and (4.3.10) by numerical integration. The results are listed here in Table II. Inspection of (4.3.7) reveals that $\frac{d\Gamma}{d\psi}$ becomes undefined when $\Gamma = 0$. When this occurs, $\varphi = \psi$. This means that the α- lines approach asymptotically a limit line. This limit line is a straight line $\psi = 0.301$ radians, the last entry of the first column of Table II.

Table II

ψ (radian)	φ	p/k	$\frac{R_\alpha}{R_{\alpha o}}$	$\frac{R_\beta}{R_{\beta o}}$
0.00	0.7854	1.0000	1.0000	1.0000
0.02	0.7654	0.9992	1.0210	0.9810
0.04	0.7453	0.9968	1.0443	0.9637
0.06	0.7250	0.9927	1.0702	0.9481
0.08	0.7044	0.9868	1.0991	0.9339
0.10	0.6833	0.9791	1.1314	0.9211
0.12	0.6617	0.9693	1.1679	0.9096
0.14	0.6393	0.9572	1.2094	0.8992
0.16	0.6160	0.9423	1.2571	0.8900
0.18	0.5914	0.9243	1.3126	0.8817
0.20	0.5652	0.9023	1.3784	0.8746
0.22	0.5367	0.8751	1.4584	0.8684
0.24	0.5048	0.8408	1.5594	0.8632
0.26	0.4678	0.7954	1.6948	0.8590
0.28	0.4207	0.7293	1.9005	0.8560
--	--	--	--	--
0.301	0.301	$-\infty$	∞	0.854

The values of ψ, $\frac{R_\alpha}{R_{\alpha o}}$ and $\frac{R_\beta}{R_{\beta o}}$ in Table II were used to draw the plastic stress field in region ABC (Fig. 4). The drawing was done with a Nestler drafting machine. $R_{\alpha o}$ for the α - line AC was chosen as R = 25 cm., the maximum radius of the conical cavity. $R_{\alpha o}$ was then incremented by 4 cm. for each successive α - line. Similar procedure was followed for the β - lines where $R_{\beta o}$ for the degenerate β - line at A was likewise R = 25 cm.. Figure 3 illustrates only that region, ABC, of the stress field calculated in the above manner that is an actual part of the true stress field for the indentation problem under analysis. Originally the plastic stress field had been determined to a larger extent as in Figure 4, but this was reduced only after the plastic stress field had been extended into the remaining region and the specific β - line through E could be determined.

4.4 Plastic Stress Field ACD.

The calculation of the plastic stress field in region AC_1D (Figure 4) is obtained from the boundary conditions in the neighbourhood of the point A and from the values of p and φ at the points of intersection of the α - characteristic AC_1 by the ψ - lines listed in Table II. These points of intersection did not coincide with the points of intersection of the α - line AC by the β - lines determined in ABC. Because p and φ are calculated values at the former points, they are used for the extension of the stress field into AC_1D. A is a singularity through which all the α - characteristics in AC_1D pass. The angular span of this fan of α - characteristics is 45^o, determined by the condition that the α - characteristic AD must meet the smooth boundary of the conical cavity at a 45^o angle. The arrangement is analogous to the first boundary - value problem [Hill, 1950] in plane strain plasticity.

Consider β - characteristic PQ of length s in Figure 4. The values of p and φ along this curve are governed by (3.3.10); namely,

$$dp - 2kd\varphi - k(\sin\varphi + \cos\varphi)\frac{dS_\beta}{r} = 0 .$$

Integration of this equation along PQ is represented by

$$\int_{P_p}^{P_Q} dp - 2k\int_{\varphi_p}^{\varphi_Q} d\varphi - k\int_0^s (\sin\varphi + \cos\varphi)\frac{dS_\beta}{r} = 0 .$$

But $dr = -\sin\varphi\, dS_\beta$ for β - characteristics. Therefore

$$p_Q - p_P - 2k(\varphi_Q - \varphi_P) + k\int_{r_P}^{r_Q}(1 + \cot\varphi)\frac{dr}{r} = 0. \qquad (4.4.1)$$

If PQ is allowed to become indefinitely small by approaching A, then $r_p \to r_R \to R$. But P_Q, P_p, φ_Q and φ_p will each have distinct values denoted as P_A', P_A, φ'_A and φ_A respectively.

Now $\lim\limits_{r \to R} \int_{r_p}^{r_Q}(1 + \cot\varphi)\frac{dr}{r} = 0$ since $1 + \cot\varphi$ is bounded for all values of φ, $(0.301 < \varphi \leq \pi/2)$, in the region ACD. In the limiting form, (4.4.1) becomes

$$P_A' - 2k\varphi_A' = P_A - 2k\varphi_A$$

or

$$\frac{P_A'}{k} = \frac{P_A}{k} + 2\Delta\varphi_A' , \quad \Delta\varphi_A' = \varphi_A' - \varphi_A . \qquad (4.4.2)$$

Since Q can arbitrarily be any point Q_i on β - characteristic in AC_1D , (4.4.2) may be written

$$\frac{P_{A\ i}'}{k} = \frac{P_A}{k} + 2\Delta\varphi_{A\ i}' , \quad \Delta\varphi_{A\ i}' = \varphi_{A\ i}' - \varphi_A . \qquad (4.4.3)$$

From Table II, $\frac{P_A}{k} = 1$ and $\varphi_A = 0.7854$. Table III is a listing of

$\frac{P'_{A_i}}{k}$ and φ'_{A_i} for 9 α - characteristics at A where $\Delta\varphi_A' = 5^o$ and is incremented 5^o for each successive α - characteristic.

Table III

i	$\Delta\varphi_{A\ i}'$	$\frac{P_{A\ i}'}{k}$	φ_{A_i}'
1	.08727	1.1745	0.8727
2	.1745	1.3491	0.9599
3	.2618	1.5236	1.0472
4	.3491	1.6981	1.1345
5	.4363	1.8727	1.2217
6	.5236	2.0472	1.3090
7	.6109	2.2217	1.3963
8	.6981	2.3963	1.4835
9	.7854	2.5708	1.5708

With reference to Figure 6, consider known point $Q(r_Q, z_Q)$ on a β - characteristic and known point $P(r_P, z_P)$ on an α - characteristic. Also both p and φ are assumed known at P and Q. To calculate $R(r,z)$, the point of intersection of the β - characteristic through P and the α - characteristic through Q, the following procedure was adopted. A first approximation of the actual location of $R(r,z)$ is made by $R_1(r_1, z_1)$ which is the point of intersection of the tangent to the β - line through P and the tangent to the α - line through Q. Both r_1 and z_1 are calculable from the simultaneous equations

$$z_1 - z_P = (r_P - r_1) \cot \varphi_P , \tag{4.4.4}$$

and $$z_1 - z_Q = (r_1 - r_Q) \tan \varphi_Q .$$

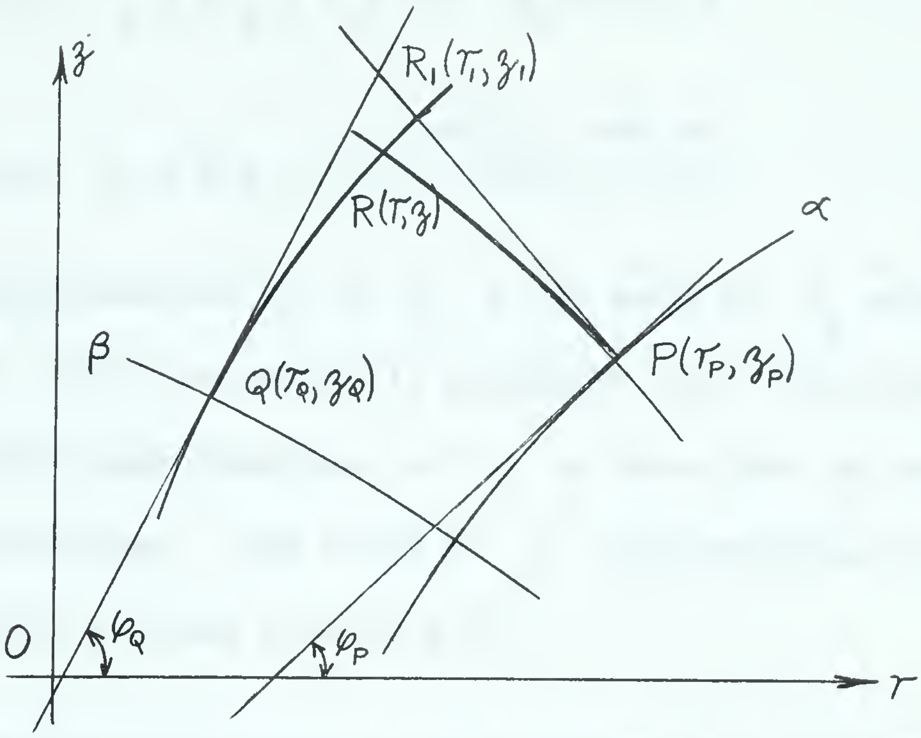


Figure 6

Intersection of Characteristics Through

Neighbouring Points to Show Approximation R_1 for True Intersection R.

The state of stress at $R(r,z)$ is governed by (3.3.8) and (3.3.10). Expressed in finite difference form these are

$$\frac{1}{k}(p_1 - p_Q) + 2(\varphi_1 - \varphi_Q) = -2\,\frac{(r_1 - r_Q + z_1 - z_Q)}{r_1 + r_Q} \qquad (4.4.5)$$

and

$$\frac{1}{k}(p_1 - p_P) - 2(\varphi_1 - \varphi_P) = -2\,\frac{(r_1 - r_P - z_1 + z_P)}{r_1 + r_P} \qquad (4.4.6)$$

respectively. An approximation for φ at R is made by φ_1 at R_1 solved from (4.4.5) and (4.4.6). A closer approximation to R is made by $R_2(r_2, z_2)$ which is the point of intersection of the straight line through Q whose slope is the mean of the slopes of the α - lines at R_1 and Q and the straight line through P whose slope is the mean of the slopes of the β - lines through R_1 and P. Both r_2 and z_2 are determined from the equations

$$r_2 - r_Q = (z_2 - z_Q) \frac{\cot \varphi_Q + \cot \varphi_1}{2} \tag{4.4.7}$$

and

$$z_2 - z_P = (r_P - r_2) \frac{\cot \varphi_P + \cot \varphi_1}{2} \tag{4.4.8}$$

A second approximation to φ at R is made by φ_z solved from (4.4.5) and (4.4.6). This procedure is continued until the difference between two successive approximations of φ is less than the nominal accuracy of the calculations. The value of p corresponding to the final value of φ is then derived from (4.4.5).

The above procedure was programmed in Fortran for the I.B.M. 1620 digital computer. The p and φ values for the 9 α - characteristics at the singular point A (Table III) and for the 14 points along the α - characteristic AC (Table II) were used to determine the fan-shaped plastic stress field. The program together with the values of p, φ, r and z at each point computed are found in Appendix I. Figure 3, however, contains only that portion of the field that is applicable to the indentation problem considered.

4.5 Plastic Stress Field ADE.

The boundary condition of zero shearing stress along the straight-line, smooth boundary AE of region ADE (Fig. 3 or 4) causes the α and β - characteristics to meet this boundary at 45° angles. Together with the α - characteristic AD, this boundary condition determines the plastic stress field in ADE. This arrangement is analogous to the third boundary-value problem [Hill, 1950] in plane strain plasticity.

With reference to Figure 4, consider point T on α - line AD at which p and φ are known. J(r,z), which is the point of intersection of the β - line through T and the straight-line boundary is sufficiently

located in the following manner. If T is in the immediate vicinity of A, then an adequate approximation to J is made by $J_1(r_1,z_1)$, the point of intersection of the boundary AE and the straight line through T whose slope is the mean of the slopes of the β - lines at T and J. Both r_1 and z_1 are determined from the equations

$$z_1 = R - r_1$$

$$\text{and} \qquad z_T - z_1 = (r_1 - r_T) \cot\left[\frac{\varphi_T+\varphi_J}{2}\right] , \tag{4.5.1}$$

where R is the radius OA of the cavity at surface. The pressure p at J_1 is determined by (4.4.6). The α - characteristic J_1K is then determined from the data on α - characteristic AD and at J_1 by the procedure described in 4.4. The remainder of the field in ADE is determined by repeating the above procedure upon successive replacement of the role of AD by J_1K.

The Fortran program for the procedure was run on an I.B.M. 1620 digital computer using the data for the 14 points on AD obtained in 4.4. Originally the field was overdetermined and it also became evident from analysis of the data that the arc length corresponding to the role of TA was too large at some stage in the computation. This rendered the approximation of the point on AE symbolized by J by the point symbolized by J_1 inadequate. A more dense distribution of points was then taken along the 7th α - line along AE. To accomplish this, large scale graphs were constructed of the variation of p, φ and z versus r along this α - line. From these graphs, 17 points were taken and another program was run on the digital computer. Doing this served the dual purpose of retaining the accuracy of the data and, also, in aiding the determination of the β - characteristic EN which passed through apex point E. This

β - characteristic EN was determined by interpolation. Graphs of the variation of φ and z versus r along MN produced were constructed. The location of N was determined by trial and error with use of equation (4.5.1). The value of p at E was then calculated as 10.210 k using (4.4.6). Figure 3 shows only that portion of the plastic stress field computed from the data that is applicable to the indentation problem. The Fortran program and data for the above procedure are found in Appendix I.

4.6 Calculation of Indentation Stress from Slip-Line Field of the Incomplete Solution.

Figure 3 is the plastic stress field or equivalently the slip-line field constructed from the data of Appendix I and represents the incipient plastic flow state of the material. Shown also in Figure 3 are values of the normal stress σ_n in units of k at 13 points along AE. These represent the distribution of the normal stresses that must be supplied by the conical indenter to produce this incipient plastic flow state. The uniform stress σ_z to be applied to the conical indenter in the axial direction and to be distributed over its base of radius R is referred to in this thesis as the indentation stress.

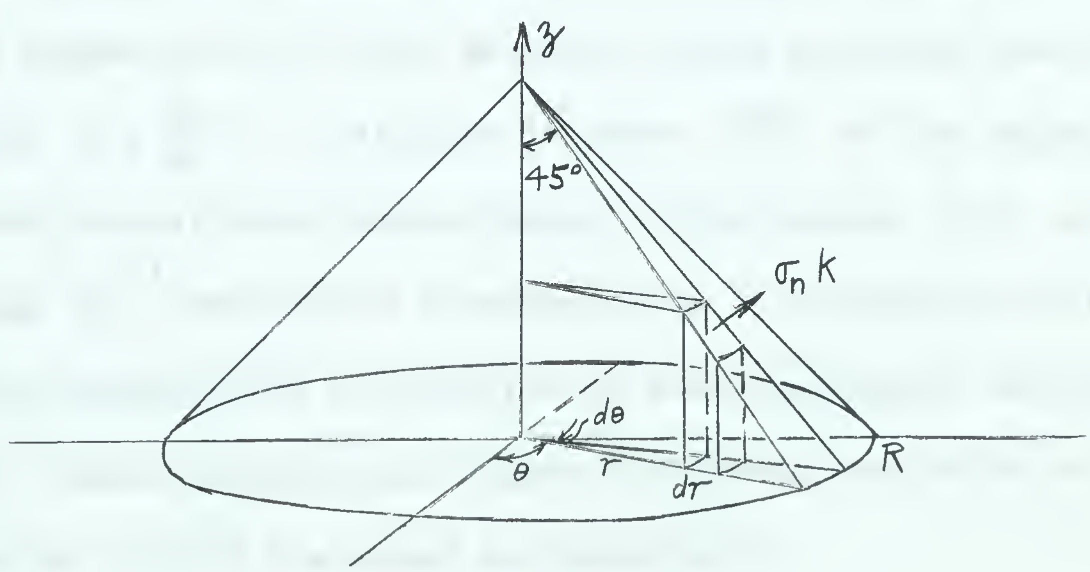


Figure 7

Reference Diagram for Computation of the Indentation Stress

With reference to Figure 7, the indentation stress σ_z is given by

$$\sigma_z = \frac{2k}{R^2} \int_0^R \sigma_n \, r \, dr \quad ,$$

which for the specific value of $R = 25$ becomes

$$\sigma_z = \frac{2k}{625} \int_0^{25} y(r) \, dr \tag{4.6.1}$$

where $y(r) = \sigma_n \, r$. The definite integral in (4.6.1) was evaluated using the Gauss quadratic formula (National Physical Laboratory [1961]). This formula states that

$$\int_a^b y(r) \, dr = \frac{1}{2} (b-a) \int_{-1}^{1} y(X) \, dX$$

$$= \frac{1}{2} (b-a) \sum_{r=1}^{n} w_r^{(n)} \; y(X_r^{(n)}) \quad ,$$

where $r = \frac{1}{2}(b+a) + \frac{1}{2}(b-a) X$, $X_r^{(n)}$ are zeros of Legendre polynomials and $w_r^{(n)}$ are weights such that the formula is exact when y is any polynomial of degree less than $2n$. Values for $X_r^{(n)}$ and $w_r^{(n)}$ were available for $n = 16$ from the University of Alberta Computing Center. However, the points along AE (Fig. 3) at which σ_n is known were not equally spaced nor did their abscissa values coincide under the transformation $X = \frac{2r}{25} - 1$ with the 16 zeros $X_r^{(n)}$ of the Legendre polynomials. This thus necessitated interpolation of the values $y(r)$ at points having abscissas $X_r^{(n)}$ under this transformation. Interpolation with unequal intervals required the calculation of some Lagrangian coefficients (Comrie [1959]). These calculations together with the evaluation of the definite integral in (4.6.1) are found in Appendix II.

Results of the above computations gave a value of $4.6424\ k$ for the indentation stress σ_z . Consequently, the yield point load L, that is, the load necessary to produce incipient plastic flow in axially symmetric identation by a conical punch of semi-angle 45^o, is given by

$$L = 4.6424\ \pi\ k\ R^2$$

where k is the maximum shearing stress of the indented material and R is the radius of the conical cavity at the surface.

4.7 Discussion and Future Requirements.

The value obtained for the yield point load has as yet not been proven to be the actual yield load. For on the basis of limit analysis theorems, the plastic stress field is unsuited for the establishment of a bound on the actual yield point load. The field has not been extended into the rigid region and, hence, not shown to be statically admissible; nor has any attempt been made to show that an associated kinematically admissible velocity field, compatible with the stress field, exists in the deforming region.

Shields [1955], in his investigations on axially symmetric indentation by a cylindrical punch, has detailed a procedure for extending the plastic stress field into the rigid zone. This was done using a method which is an application of the theorems of Bishop [1953] on the admissibility of incomplete solutions. It is proposed that a direct application of Shield's procedure to the present problem in conical indentation should result in the extension of the plastic stress field into the rigid zone. Moreover, the resulting stress free surface should lie wholly within the boundaries of the indented material, thus meeting one of the requirements of Theorem 1 of Bishop's. By this theorem, if for a fully

plastic continuation of the stress solution of a fully plastic region which is in equilibrium with an applied external load and which contains a compatible deformation mode, the resulting stress free surface lies wholly within or upon the external boundary of the material, then a complete solution exists and the actual yield-point load has been found.

To meet the other requirement of the theorem, a velocity field must be found in the deforming region compatible with the plastic stress field. A method by which this may be done is now considered. The region ABCDE (Fig. 3) is deforming, whereas, the region above it is rigid. Thus there is a possible velocity discontinuity across the β-line BCDE. In terms of the velocity resolutes U and W defined by equations (3.3.18), it is possible to show that the boundary conditions to be satisfied on this β-line is $U = W = 0$. If now the boundary condition along the non-characteristic line AE can be established when the punch is penetrating in an axial direction at a rate w_o, say, the entire velocity field can be determined by a method analogous to that type in plane strain (type (iii), Hill [1950]).

In accord with the above discussion, the validity of the value $4.6424\pi kR^2$ for the yield point load thus rests on the determination of the kinematically admissible velocity field. Until this has been done, this value is only a tentative one.

APPENDIX I

COMPUTER PROGRAMS AND OUTPUT DATA FOR PLASTIC STRESS FIELDS

The calculations for the individual plastic stress fields which form the region ABCDE (Fig. 3) were greatly facilitated by the use of an I.B.M. 1620 digital computer. The iterative processes by which these calculations had to be made were ideally suited as computer problems. The Fortran source programs for each field were designed so as to be adaptable to other axially symmetric indentation problems.

A. Plastic Stress Field AC_1D

1. Procedure

As detailed in 4.4, Chapter IV, the actual location of $R(r,z)$ in Figure 6 is approximated firstly by $R_1(r_1,z_1)$. The coordinates of this latter point are

$$z_1 = \frac{\{(r_P - r_Q) + z_Q \cot \varphi_Q\} \cot \varphi_P + z_P}{\cot \varphi_Q \cot \varphi_P + 1} \tag{A.1.1}$$

and $$r_1 = (z_1 - z_Q) \cot \varphi_Q + r_Q \ ,$$

as derived from equations (4.4.4). Now the state of stress at $R(r,z)$ is governed by the finite difference equations (4.4.5) and (4.4.6). Solving these equations for φ yields

$$\varphi = \frac{\varphi_P + \varphi_Q}{2} - \frac{p_P - p_Q}{4} + \frac{1}{2}\left\{\frac{(r-z)-(r_P - z_P)}{r + r_P} - \frac{(r+z)-(r_Q+z_Q)}{r + r_Q}\right\}. \tag{A.1.2}$$

φ cannot be determined explicitly from equation (A.1.2) as r and z are as yet unknown. However, an approximation to φ is attained by say φ_1 upon giving the values of r_1 and z_1 to r and z respectively. Then a second approximation to $R(r,z)$ is made by $R_2(r_2,z_2)$ whose coordinates are given by

$$z_2 = \frac{\left(\frac{\cot\varphi_Q+\cot\varphi_1}{2}\right)z_Q + \left(\frac{2}{\cot\varphi_P+\cot\varphi_1}\right)z_P + r_P - r_Q}{\frac{\cot\varphi_Q+\cot\varphi_1}{2} + \frac{2}{\cot\varphi_P+\cot\varphi_1}} \tag{A.1.3}$$

and $$r_2 = (z_2 - z_Q)\left(\frac{\cot\varphi_Q+\cot\varphi_1}{2}\right) + r_Q ,$$

as derived from equations (4.4.7) and (4.4.8). Upon substitution of these values of r_2 and z_2 for r and z in (A.1.2), a second approximation to φ is attained. This procedure is continued until the difference between two successive approximations of φ is less than the nomial accuracy of the calculations. The coordinates of $R(r,z)$ are then taken as the final values of (A.1.3) which attained this condition. Finally, p at $R(r,z)$ is solved using equation (4.4.5).

The above procedure is concerned only with the location and state of stress of one point $R(r,z)$. However, it is possible to determine an entire β-line (or α-line) easily with a computer by transferring the role of points and introducing others. For at singular point A (Fig. 4), which is to be considered as a degenerate β-line, there is initially the first of 9 points (Table III) to play the role of Q and each of 14 points (Table II) along AC_1 to play the role of P. After $R(r,z)$ has been calculated, it is transferred to act as P and the next point at A is introduced to act as Q. In this manner the first α-line in AC_1D is determined. The arrangement then becomes similar to the original one with the newly computed α-line acting as AC_1. By repeating the entire procedure the stress field is determined.

2. Fortran Source Program, Input and Output Data.

The equationsof A.1 were translated into Fortran language of the computer using the following notation for the various quantities

appearing in them:

$$X(I) = r_Q\,, \quad Y(I) = z_Q\,, \quad T(I) = \varphi_Q\,, \quad P(I) = p_Q$$

$$XX = r_P\,, \quad YY = z_P\,, \quad TT = \varphi_P\,, \quad PP = p_P$$

$$XXX = r_1\,, \quad YYY = z_1$$

$$XXXX = r_2\,, \quad YYYY = z_2$$

$$w = \cot\varphi_Q\,, \quad v = \cot\varphi_P$$

$$B = \frac{(r-z)-(r_Q-z_Q)}{r+r_Q}\,, \qquad C = \frac{(r+z)-(r_P+z_P)}{r+r_P}$$

$$TA = \frac{\cot\varphi_Q+\cot\varphi_1}{2}\,, \qquad TB = \frac{2}{\cot\varphi_P+\cot\varphi_1}\,.$$

The Fortran source program, which embodies the procedure of A.1, is found in the following attached sheet together with the input data which gives the locations and state of stress at the known points along AC_1 and the singular point A. These are followed by the output data which determines the fan-shaped plastic stress field AC_1D.

B. Plastic Stress Field ADE.

For the remaining region ADE (Fig. 4) the procedure adopted was essentially the same as the one used above except that successive α-lines rather than β-lines were computed to extend the stress field AC_1D into the region. For this reason detailed discussion of the procedure would be unwarranted. Thus only the Fortran source statements and the input and output data for ADE are to be found in the following attached sheets.

Fortran Source Statement for Plastic Stress Field AC_1D

```
      DIMENSION X(50),Y(50),T(50),P(50)
2     READ,N,M,JJ,E
      DO 1  I=1,N
1     READ,X(I),Y(I),T(I),P(I)
      DO 20  J=JJ,M
      READ,XX,YY,TT,PP
      DO 30  I=1,N
      V=0.0
      W=COS(T(I))/SIN(T(I))
      U=COS(TT)/SIN(TT)
      YYY=(((XX-X(I))+Y(I)*W)*U+YY)/(U*W+1.0)
      XXX=(YYY-Y(I))*W+X(I)
5     B=(XXX-YYY-XX+YY)/(XXX+XX)
      C=(XXX+YYY-X(I)-Y(I))/(XXX+X(I))
      PHI=(T(I)+TT)/2.0-(PP-P(I))/4.0+(B-C)/2.0
      F=ABS(V-PHI)
      IF (F-E)7,7,8
8     V=PHI
      TA=(W+COS(V)/SIN(V))/2.0
      TB=2.0/(U+COS(V)/SIN(V))
      YYYY=(TA*Y(I)+TB*YY+XX-X(I))/(TA+TB)
      XXXX=(YYYY-Y(I))*TA+X(I)
      YYY=YYYY
      XXX=XXXX
      GO TO 5
7     PF=(T(I)-C-PHI)*2.0+P(I)
      IF (SENSE SWITCH 3)9,10
9     PRINT,I,J,PF,PHI,XXX,YYY
10    PUNCH,I,J,PF,PHI
      PUNCH,I,J,XXX,YYY
      X(I)=XXX
      Y(I)=YYY
      T(I)=PHI
      P(I)=PF
      XX=XXX
      YY=YYY
      TT=PHI
30    PP=PF
20    CONTINUE
      PAUSE
      GO TO 2
      END
```

Input Data for Plastic Stress Field AC_1D

9 14 1 0.00001

Data at Singular Point A

25.0000	0.0000	0.8727	1.1745
25.0000	0.0000	0.9599	1.3491
25.0000	0.0000	1.0472	1.5236
25.0000	0.0000	1.1345	1.6981
25.0000	0.0000	1.2217	1.8727
25.0000	0.0000	1.3090	2.0472
25.0000	0.0000	1.3963	2.2217
25.0000	0.0000	1.4835	2.3963
25.0000	0.0000	1.5708	2.5708

Data for α-characteristic AC_1

25.5199	0.5105	0.7654	0.9992
26.0866	1.0440	0.7453	0.9968
26.7068	1.6042	0.7259	0.9927
27.3896	2.1957	0.7044	0.9868
28.1436	2.8237	0.6833	0.9791
28.9876	3.4552	0.6617	0.9693
29.9393	4.2190	0.6393	0.9572
31.0262	5.0070	0.6160	0.9423
32.2847	5.8749	0.5914	0.9243
33.7732	6.8462	0.5652	0.9023
35.5813	7.9567	0.5367	0.8751
37.8677	9.2667	0.5048	0.8409
40.9459	10.8925	0.4670	0.7954
45.6624	13.1306	0.4207	0.7293

OUTPUT DATA

α-line	β-line		
1	1	p= 1.1757091	φ= .62161860
1	1	r= 25.474588	z= .59150521
2	1	1.3419911	.44827975
2	1	25.426318	.59301509
3	1	1.5265398	1.0245476
3	1	25.372868	.61885572
4	1	1.7052694	1.1115096
4	1	25.316862	.65596341
5	1	1.8825932	1.1982410
5	1	25.256645	.68595927
6	1	2.0565770	1.2850912
6	1	25.198272	.70599790
7	1	2.2359746	1.3720617
7	1	25.136223	.72155685
8	1	2.4129740	1.4591000
8	1	25.073010	.73126318
9	1	2.5900606	1.5462792
9	1	25.009018	.73504251
1	2	1.1747827	.61017245
1	2	25.994892	1.1597563
2	2	1.3530607	.91656585
2	2	25.834912	1.2199505
3	2	1.5516826	1.0021444
3	2	25.767724	1.2949258
4	2	1.7105712	1.0880509
4	2	25.674105	1.3609595
5	2	1.8896474	1.1742110
5	2	25.554897	1.4111947
6	2	2.0687856	1.2605410
6	2	25.430819	1.4850646
7	2	2.2485024	1.3470906
7	2	25.302809	1.4981405
8	2	2.4281105	1.4356264
8	2	25.171897	1.5228675
9	2	2.6078794	1.5200117
9	2	25.038797	1.5545657
1	3	1.1722829	.60945412
1	3	26.567478	1.7492149
2	3	1.3523402	.88405800
2	3	26.415558	1.0026171
3	3	1.5329180	.97899306
3	3	26.252098	2.0052079
4	3	1.7139326	1.0641574
4	3	26.078160	2.1099245
5	3	1.8952962	1.1495269
5	3	25.894972	2.2018005
6	3	2.0770780	1.2352255
6	3	25.703579	2.2780559
7	3	2.2591798	1.3212460
7	3	25.505346	2.3378547
8	3	2.4414950	1.4075514
8	3	25.301729	2.3806014
9	3	2.6240904	1.4942291
9	3	25.094000	2.4055769
1	4	1.1678921	.78775593
1	4	27.201459	2.4000190
2	4	1.3496555	.87150110
2	4	26.996050	2.5000108
3	4	1.5521095	.95527414

3	4	26.774263	2.7003342
4	4	1.7151570	1.0591007
4	4	26.457972	2.7147111
5	4	1.8958288	1.1240405
5	4	26.250915	3.0475005
6	4	2.0830628	1.2089889
6	4	26.024237	3.1597286
7	4	2.2677968	1.2943330
7	4	25.751077	3.2503716
8	4	2.4525214	1.3800833
8	4	25.469951	3.3175977
9	4	2.6384622	1.4665101
9	4	25.180691	3.3605337
1	5	1.1616168	.76552985
1	5	27.906510	3.0926565
2	5	1.3449518	.84800346
2	5	27.646097	3.3429876
3	5	1.5292119	.93052800
3	5	27.363799	3.5727881
4	5	1.7143136	1.0140135
4	5	27.061177	3.7794040
5	5	1.9001696	1.0979900
5	5	26.739965	3.9615901
6	5	2.0868462	1.1818301
6	5	26.401790	4.1175178
7	5	2.2742406	1.2661501
7	5	26.048670	4.2454356
8	5	2.4622416	1.3511838
8	5	25.682923	4.3437956
9	5	2.6508920	1.4368088
9	5	25.306581	4.4111217
1	6	1.1530524	.74265190
1	6	28.659404	3.8931417
2	6	1.3378554	.82358934
2	6	28.381934	4.1570543
3	6	1.5238062	.90550507
3	6	28.038531	4.4524003
4	6	1.7108512	.98752510
4	6	27.664838	4.7209176
5	6	1.8988496	1.0699734
5	6	27.268807	4.9599662
6	6	2.0879380	1.1529475
6	6	26.855163	5.1674808
7	6	2.2780016	1.2364778
7	6	26.411242	5.3411508
8	6	2.4689346	1.3205963
8	6	25.954633	5.4787982
9	6	2.6607874	1.4054055
9	6	25.442688	5.5714688
1	7	1.1420779	.71885040
1	7	29.598549	4.6425058
2	7	1.3282051	.79579935
2	7	29.221864	5.0414397
3	7	1.5157135	.87704641
3	7	28.810426	5.4121152
4	7	1.7045420	.95917109
4	7	28.368036	5.7515100
5	7	1.8946210	1.0409144
5	7	27.894084	6.0379128

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4	9	1.6810531	[illegible]
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9	11	27.932831	14.358215
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4	13	[illegible]	[illegible]
4	13	[illegible]	[illegible]
5	13	[illegible]	[illegible]
5	13	[illegible]	[illegible]
6	13	[illegible]	[illegible]
6	13	35.177771	[illegible]
7	13	[illegible]	[illegible]
7	13	[illegible]	[illegible]
8	13	[illegible]	[illegible]
8	13	32.480804	20.092868
9	13	[illegible]	[illegible]
9	13	30.989624	[illegible]
1	14	[illegible]	[illegible]
1	14	44.859458	[illegible]

2	14	[illegible]	[illegible]
2	14	[illegible]	[illegible]
3	14	[illegible]	[illegible]
3	14	[illegible]	[illegible]
4	14	[illegible]	[illegible]
4	14	[illegible]	[illegible]
5	14	[illegible]	[illegible]
5	14	[illegible]	[illegible]
6	14	1.8689487	[illegible]
6	14	[illegible]	22.461616
7	14	[illegible]	[illegible]
7	14	37.571482	[illegible]
8	14	[illegible]	[illegible]
8	14	[illegible]	[illegible]
9	14	[illegible]	[illegible]
9	14	[illegible]	[illegible]

FORTRAN SOURCE STATEMENT FOR PLASTIC STRESS FIELD ADE

```
      DIMENSION X(50),Y(50),P(50),T(50)
    2 READ,N,M,JJ,K,F
      DO 1 I=1,M
      READ,P(I),T(I)
    1 READ,X(I),Y(I)
      DO 20 J=JJ,N
      TT=T(1)
      T2=1.5708
      T(1)=(T(1)+T2)/2.0
   17 U=COS(T(1))/SIN(T(1))
      X2=(Y(1)+X(1)*U-25.0)/(U-1.0)
      Y2=25.0-X2
   18 T(1)=TT
      R=(X2-Y2-X(1)+Y(1))*2.0/(X2+X(1))
      P2=(T2-T(1))*2.0+P(1)-R
      IF (SENSE SWITCH 3)6,7
    6 PRINT,J,K,X2,Y2,P2,T2
    7 PUNCH,J,K,P2,T2
      PUNCH,J,K,X2,Y2
      DO 15 I=2,M
      V=0.0
      S=COS(T2)/SIN(T2)
      R=COS(T(I))/SIN(T(I))
      Y3=((X(I)-X2+Y2*S)*R+Y(I))/(S*R+1.0)
      X3=(Y3-Y2)*S+X2
   10 D=(X3-Y3-X(I)+Y(I))/(X3+X(I))
      C=(X3+Y3-X2-Y2)/(X3+X2)
      T3=(T2+T(I))/2.0-(P(I)-P2)/4.0+(D-C)/2.0
      E=ABS(V-T3)
      IF (E-F)8,8,9
    9 V=T3
      RA=(S+COS(V)/SIN(V))/2.0
      RB=2.0/(R+COS(V)/SIN(V))
      Y4=(RA*Y2+RB*Y(I)+X(I)-X2)/(RA+RB)
      X4=(Y4-Y2)*RA+X2
      Y3=Y4
      X3=X4
      GO TO 10
    8 P3=(T2-C-T3)*2.0+P2
      IF (SENSE SWITCH 3)11,12
   11 PRINT,J,I,X3,Y3,P3,T3
   12 PUNCH,J,I,X3,Y3
      PUNCH,J,I,P3,T3
      X2=X3
      Y2=Y3
      P2=P3
      T2=T3
      X(I-1)=X3
      Y(I-1)=Y3
      P(I-1)=P3
   15 T(I-1)=T3
   20 CONTINUE
      PAUSE
      GO TO 2
      END
```

INPUT DATA FOR PLASTIC STRESS FIELD ADE

α-line	β-line		
9	1	p=[illegible]	φ=[illegible]
9	1	r=[illegible]	z=[illegible]
9	2	2.60[illegible]4	1.[illegible]
9	2	25.03[illegible]	1.[illegible]
9	3	2.6240[illegible]4	1.4[illegible]
9	3	25.[illegible]4[illegible]	[illegible].4[illegible]
9	4	2.63[illegible]48[illegible]	1.46[illegible]
9	4	2[illegible].1[illegible]6[illegible]1	[illegible].365[illegible]
9	5	2.6[illegible]	1.436[illegible]
9	5	2[illegible].306[illegible]	[illegible].4[illegible]
9	6	2.660[illegible]74	1.4[illegible]4[illegible]
9	6	2[illegible].48268[illegible]	[illegible].[illegible]764[illegible]
9	7	2.66776[illegible]	1.3717026
9	7	25.7235[illegible]	6.[illegible]48[illegible]1[illegible]
9	8	2.67[illegible]16[illegible]	1.3[illegible]4
9	8	26.04[illegible]72[illegible]	8.[illegible]6[illegible]24[illegible]
9	9	2.669[illegible]4	1.2[illegible]4[illegible]76
9	9	26.4[illegible]1146	10.[illegible]
9	10	2.6614754	1.2[illegible]041
9	10	27.0[illegible]3[illegible]4	12.01[illegible]44[illegible]
9	11	2.644[illegible]7[illegible]2	1.2020419
9	11	27.[illegible]32[illegible]31	14.[illegible]
9	12	2.61[illegible]723[illegible]	1.14[illegible]142
9	12	29.141487	17.2[illegible]61[illegible]
9	13	2.5663156	1.[illegible]77[illegible]044
9	13	30.9[illegible]624	20.962[illegible]
9	14	2.478543[illegible]	.9[illegible]
9	14	34.211437	26.34[illegible]69[illegible]

OUTPUT DATA FOR PLASTIC STRESS FIELD ADE

α-line	β-line	r	z	p	φ
1	1	24.255116	.74488400	2.6700830	1.5708000
1	2	24.263818	1.5640013	2.6881427	1.5446600
1	3	24.301383	2.4572810	2.7045221	1.5173[illegible]
1	4	24.367902	3.4366166	2.7189905	1.4886307
1	5	24.473202	4.5143763	2.7313561	1.4582147
1	6	24.628340	5.7121557	2.7410509	1.4258245
1	7	24.648038	7.0527746	2.7476271	1.3[illegible]
1	8	25.153235	8.5685163	2.7500981	1.3533466
1	9	25.574217	10.302651	2.7475369	1.3121430
1	10	26.157828	12.320157	2.7381103	1.2665750
1	11	26.970745	14.711267	2.7103391	1.2153447
1	12	28.175567	17.667646	2.6871039	1.1563244
1	13	30.018171	21.480236	2.6322530	1.0860717
1	14	33.249122	26.079370	2.5397225	.[illegible]
2	1	23.425014	1.5740860	2.7761267	1.5708000
2	2	23.437935	2.4925362	2.7927407	1.5436370
2	3	23.481232	3.4987907	2.8073335	1.5120722
2	4	23.562882	4.6065980	2.8196787	1.4815159
2	5	23.694030	5.8380428	2.8291625	1.4470402
2	6	23.889684	7.2167072	2.8352771	1.4117077
2	7	24.171155	8.7755923	2.8369555	1.3726663
2	8	24.569349	10.559069	2.8331623	1.3287030
2	9	25.131978	12.633384	2.8219147	1.2823780
2	10	25.936134	15.098581	2.8005231	1.2290464
2	11	27.110742	18.125041	2.7645511	1.167[illegible]

2	12	28.959766	22.031740	2.7058339	1.0048167
2	13	32.206993	27.650066	2.6027195	1.0011915
2	14	32.207392	27.650690	2.6026976	1.0011767
3	1	22.494176	2.5058240	2.8907371	1.5708005
3	2	22.510121	3.5418467	2.9055395	1.5400840
3	3	22.563954	4.6830567	2.9179181	1.5071211
3	4	22.666964	5.9521767	2.9272015	1.4723442
3	5	22.834547	7.3735947	2.9328039	1.4346211
3	6	23.088530	8.9811441	2.9335561	1.3937092
3	7	23.460652	10.820416	2.9292645	1.3496396
3	8	23.999803	12.959016	2.9148180	1.2971506
3	9	24.784950	15.408862	2.8902071	1.2433000
3	10	25.957050	18.613072	2.8436265	1.1751175
3	11	27.798299	22.624391	2.7833142	1.1021495
3	12	31.071035	28.373621	2.6713677	1.0061192
3	13	31.071434	28.374252	2.6713449	1.0061141
3	14	31.072801	28.376411	2.6712621	1.0060398
4	1	21.441714	3.5582860	3.0164567	1.5708000
4	2	21.461877	4.7369361	3.0289709	1.5365064
4	3	21.530883	6.0486170	3.0381149	1.4999148
4	4	21.664657	7.5186734	3.0431737	1.4602556
4	5	21.883755	9.1819078	3.0428509	1.4171482
4	6	22.227056	11.085290	3.0357015	1.3697330
4	7	22.739076	13.297827	3.0105803	1.3173067
4	8	23.503313	15.923596	2.9909297	1.2583241
4	9	24.664736	19.138121	2.9445376	1.1906787
4	10	26.512313	23.267564	2.8704285	1.1100747

4	11	29.822020	29.160995	2.7457621	1.0098887
4	12	29.823320	29.161633	2.7457391	1.0098863
4	13	29.824691	29.163816	2.7456515	1.0098681
4	14	29.827847	29.168840	2.7454419	1.0098754
5	1	20.242206	4.7577940	3.1568730	1.5708000
5	2	20.268601	6.1170272	3.1660130	1.5220269
5	3	20.360060	7.6439472	3.1705580	1.4809222
5	4	20.540189	9.3718435	3.1600206	1.4440413
5	5	20.843471	11.350189	3.1597942	1.3935210
5	6	21.322725	13.649677	3.1401304	1.3374593
5	7	22.062957	16.376142	3.1062970	1.2744650
5	8	23.214565	19.707721	3.0523794	1.2024081
5	9	25.075403	23.972962	2.9675440	1.1170664
5	10	28.440739	30.027203	2.8273062	1.0120716
5	11	28.441140	30.027846	2.8272806	1.0120661
5	12	28.442521	30.030055	2.8271890	1.0120489
5	13	28.445695	30.035133	2.8269678	1.0120144
5	14	28.451976	30.045178	2.8265074	1.0119577
6	1	18.854647	6.1453530	3.3172832	1.5708000
6	2	18.890642	7.7367351	3.3214834	1.5275852
6	3	19.017188	9.5412406	3.3186388	1.4760660
6	4	19.271139	11.609406	3.3067724	1.4273590
6	5	19.708741	14.013726	3.2826104	1.3605996
6	6	20.420114	16.861929	3.2419154	1.2821655
6	7	21.563276	20.333926	3.1779419	1.2142781
6	8	23.448045	24.779962	3.0795590	1.1237374
6	9	26.893233	30.994923	2.9177430	1.0118966

6	10	26.893636	30.[illegible]68	3.0177[illegible]	1.01[illegible]
6	11	26.895035	30.[illegible]780[illegible]	3.0176178	1.011810[illegible]
6	12	26.898245	31.002[illegible]	3.0173814	1.011[illegible]
6	13	26.904586	31.013072	3.0168[illegible]	1.0117[illegible]
6	14	26.916515	31.032137	3.015[illegible]	1.011615[illegible]
7	1	17.225622	7.[illegible]43780	3.5062008	1.5[illegible]
7	2	17.277335	9.6713206	3.5021862	1.[illegible]63242
7	3	17.462594	11.850030	3.4871042	1.4[illegible]
7	4	17.843070	14.385086	3.457[illegible]	1.[illegible]
7	5	18.516811	17.385787	3.4068618	1.[illegible]
7	6	19.653212	21.032342	3.3283612	1.[illegible]
7	7	21.578647	25.650983	3.2085872	1.[illegible]
7	8	25.142193	32.095087	3.0193468	1.00784[illegible]
7	9	25.142601	32.095735	3.0193168	1.007[illegible]
7	10	25.144031	32.098001	3.0192102	1.00781[illegible]
7	11	25.147303	32.103186	3.0189536	1.00777[illegible]
7	12	25.153755	32.113409	3.0184250	1.[illegible]
7	13	25.165872	32.132602	3.0173332	1.007[illegible]
7	14	25.188122	32.167835	3.0154282	1.0074446
8	1	15.274100	9.7259000	3.7275810	1.5[illegible]
8	2	15.354420	12.044829	3.7102519	1.5[illegible]
8	3	15.648343	14.750144	3.6813891	1.4[illegible]
8	4	16.267577	17.951716	3.6168573	1.3[illegible]
8	5	17.399894	21.826135	3.5162[illegible]	1.[illegible]
8	6	19.395643	26.688101	3.3651[illegible]	1.[illegible]
8	7	23.142100	33.271[illegible]	3.1344138	.99[illegible]
8	8	23.142[illegible]	[illegible]	3.1341[illegible]	.99[illegible]

8	9	23.143998	33.374985	3.1343816	[illegible]
8	10	23.147374	[illegible]	3.1337762	[illegible]
8	11	23.154022	33.380474	3.1321324	[illegible]
8	12	23.166484	33.409744	[illegible]	[illegible]
8	13	23.182220	33.448080	3.12[illegible]	[illegible]
8	14	23.232630	33.512075	[illegible]	[illegible]
9	1	12.869196	12.130804	4.0317914	[illegible]
9	2	13.009327	15.071225	3.9917714	[illegible]
9	3	13.533823	18.558974	3.9046624	[illegible]
9	4	14.665380	22.755265	3.7650836	[illegible]
9	5	16.700312	27.040070	3.5612352	[illegible]
9	6	20.838443	34.896734	3.2634694	[illegible]
9	7	20.838882	34.897382	3.2634307	[illegible]
9	8	20.840448	34.899693	3.2632925	[illegible]
9	9	20.844015	34.904954	3.2629640	[illegible]
9	10	20.851014	34.915275	3.2622963	[illegible]
9	11	20.864099	34.934566	3.2610063	[illegible]
9	12	20.888017	34.969814	[illegible]	[illegible]
9	13	20.933332	35.036555	3.2538080	[illegible]
9	14	21.025757	35.172523	3.2445083	[illegible]
10	1	9.7749937	15.225007	4.4702269	[illegible]
10	2	10.075958	19.177499	[illegible]	[illegible]
10	3	11.208134	[illegible]	4.1333854	1.2544694
10	4	13.609575	29.944911	3.8163444	[illegible]
10	5	18.214[illegible]6	36.771690	3.3970282	[illegible]
10	6	18.215020	36.772335	3.3969897	[illegible]
10	7	18.216744	36.774691	3.3969111	[illegible]

10	8	18.223667	36.779841	3.3964130	.02648470
10	9	18.228311	36.790022	3.3956086	.02648384
10	10	18.242376	36.804112	3.3940636	.02624516
10	11	18.268574	36.842582	3.3911712	.02604475
10	12	18.317752	36.898078	3.3856357	.02560177
10	13	18.419150	37.042276	3.3744975	[illegible]
10	14	18.659276	37.361246	3.3458707	.02202628
11	1	6.4741122	19.525888	6.2019271	1.5708000
11	2	6.5210464	25.488298	4.7986729	1.2330604
11	3	9.7635577	31.839418	4.1254073	[illegible]
11	4	15.594601	39.008886	3.4462224	.80750470
11	5	15.595240	39.009471	3.4461609	.80749891
11	6	15.597281	39.011605	3.4459376	.80747665
11	7	15.601903	39.016426	3.4454179	.80742418
11	8	15.610886	39.025824	3.4443795	.80736413
11	9	15.627590	39.043277	3.4424954	.80725744
11	10	15.657926	39.074965	3.4386739	.80700812
11	11	15.715194	39.134762	3.4317555	.80686898
11	12	15.832192	39.256830	3.4176682	.80632464
11	13	16.115087	39.551154	3.3860777	.80406206
11	14	17.050732	40.508088	3.3014076	.78833300
12	1	-1.9288851	26.928885	9.7817196	1.5708000

FORTRAN SOURCE STATEMENT FOR PLASTIC STRESS FIELD ADE

(Run 2)

```
      DIMENSION X(50),Y(50),P(50),T(50)
    2 READ,N,M,JJ,K,E
      DO 1 I=1,M
      READ,P(I),T(I)
    1 READ,X(I),Y(I)
      DO 20 J=JJ,N
      TT=T(1)
      T2=1.5708
      T(1)=(T(1)+T2)/2.0
   17 U=COS(T(1))/SIN(T(1))
      X2=(Y(1)+X(1)*U-25.0)/(U-1.0)
      Y2=25.0-X2
   18 T(1)=TT
      B=(X2-Y2-X(1)+Y(1))*2.0/(Y2+X(1))
      P2=(T2-T(1))*2.0+P(1)-B
      IF (SENSE SWITCH 3)6,7
    6 PRINT,J,K,X2,Y2,P2,T2
    7 PUNCH,J,K,P2,T2
      PUNCH,J,K,X2,Y2
      DO 15 I=2,M
      V=0.0
      S=COS(T2)/SIN(T2)
      R=COS(T(I))/SIN(T(I))
      Y3=((X(I)-X2+Y2*S)*R+Y(I))/(S*R+1.0)
      X3=(Y3-Y2)*S+X2
   10 D=(X3-Y3-X(I)+Y(I))/(X3+X(I))
      C=(X3+Y3-X2-Y2)/(Y3+X2)
      T3=(T2+T(I))/2.0-(P(I)-P2)/4.0+(D-C)/2.0
      F=ABS(V-T3)
      IF (F-E)8,8,9
    9 V=T3
      RA=(S+COS(V)/SIN(V))/2.0
      RB=2.0/(R+COS(V)/SIN(V))
      Y4=(RA*Y2+RB*Y(I)+X(I)-X2)/(RA+RB)
      X4=(Y4-Y2)*RA+X2
      Y3=Y4
      X3=X4
      GO TO 10
    8 P3=(T2-C-T3)*2.0+P2
      IF (SENSE SWITCH 3)11,12
   11 PRINT,J,I,X3,Y3,P3,T3,
   12 PUNCH,J,I,X3,Y3
      PUNCH,J,I,P3,T3
      X2=X3
      Y2=Y3
      P2=P3
      T2=T3
      X(I-1)=X3
      Y(I-1)=Y3
      P(I-1)=P3
   15 T(I-1)=T3
   20 CONTINUE
      PAUSE
      GO TO 2
      END
```

INPUT DATA FOR PLASTIC STRESS FIELD ADE (Run 2)

```
 17   17    1    1
 0.00001

p=3.5044   1.5364 =φ
r=17.2500   9.0200=z
 3.5022   1.5163
 17.2773   9.6713
 3.4964   1.4857
 17.3500  10.6400
 3.4872   1.4558
 17.4626  11.8500
 3.4724   1.4190
 17.6500  13.2000
 3.4572   1.3882
 17.8431  14.3850
 3.4453   1.3684
 18.0000  15.1800
 3.4301   1.3450
 18.2000  16.1250
 3.4154   1.3237
 18.4000  16.9550
 3.4069   1.3123
 18.5168  17.3858
 3.3900   1.2919
 18.7500  18.2200
 3.3723   1.2706
 19.0000  19.0600
 3.3585   1.2567
 19.2000  19.7200
 3.3450   1.2420
 19.4000  20.3400
 3.3385   1.2352
 19.5000  20.6400
 3.3334   1.2285
 19.6500  20.9250
 3.3284   1.2263
 19.6537  21.0323
```

OUTPUT DATA FOR PLASTIC STRESS FIELD ADE (Run 2)

α-line	β-line	r	z	p	φ
1	1	24.255116	.74488400	2.6700839	1.5708000
1	2	24.265818	1.5640013	2.6891425	1.5446697
1	3	24.301383	2.4572810	2.7045221	1.5173536
1	4	24.367902	3.4366166	2.7189905	1.4984307
1	5	24.473202	4.5143963	2.7313561	1.4592240
1	6	24.628340	5.7121557	2.7410509	1.4258245
1	7	24.848038	7.0527746	2.7476271	[illegible]
1	8	25.153239	8.5685163	2.7500981	1.3533466
1	9	25.574217	10.302651	2.7475369	1.3131430
1	10	26.157828	12.320197	2.7381193	1.2665759
1	11	26.979745	14.719267	2.7193391	1.2153447
1	12	28.175567	17.667646	2.6871099	1.1563244
1	13	30.018171	21.480236	2.6332539	1.0860717
1	14	33.249122	26.978370	2.5387225	[illegible]
2	1	23.425014	1.5749860	2.7761267	1.5708000
2	2	23.437935	2.4925362	2.7927407	[illegible]
2	3	23.481232	3.4987907	2.8073335	[illegible]
2	4	23.562882	4.6065980	2.8196787	[illegible]
2	5	23.694030	5.8380428	2.8291625	[illegible]
2	6	23.889684	7.2167072	2.8352771	1.4117977
2	7	24.171155	8.7755923	2.8369551	[illegible]
2	8	24.569349	10.559069	2.8331623	1.3200020
2	9	25.131978	12.633384	2.8219147	[illegible]
2	10	25.936124	[illegible]	2.8005231	[illegible]
2	11	27.119742	18.128041	2.7645511	[illegible]

2	12	28.959766	22.031740	2.7053539	[illegible]
2	13	32.206993	27.650066	2.6027195	[illegible]
2	14	32.207392	27.650690	2.6026975	[illegible]
3	1	22.494176	2.5058240	2.8907371	1.5708000
3	2	22.510121	3.5418467	2.9055395	1.5400240
3	3	22.563954	4.6830567	2.9172181	1.5073219
3	4	22.666964	5.9521767	2.9272015	[illegible]
3	5	22.834547	7.3735947	2.9328039	[illegible]
3	6	23.088530	8.9811441	2.9335561	[illegible]
3	7	23.460652	10.823416	2.9292845	[illegible]
3	8	23.999803	12.969016	2.9148189	[illegible]
3	9	24.784950	15.498869	2.8902071	1.2423000
3	10	25.957090	18.613072	2.8406265	[illegible]
3	11	27.798299	22.624391	2.7828143	[illegible]
3	12	31.071035	28.373621	2.6713677	[illegible]
3	13	31.071434	28.374252	2.6713449	1.0061141
3	14	31.072801	28.376411	2.6712621	[illegible]
4	1	21.441714	3.5582860	3.0164567	1.5708000
4	2	21.461877	4.7369361	3.0299729	[illegible]
4	3	21.530883	6.0496179	3.0391149	[illegible]
4	4	21.664657	7.5186734	3.0431737	[illegible]
4	5	21.889759	9.1919078	3.0429[illegible]	[illegible]
4	6	22.227056	11.085289	3.0357915	[illegible]
4	7	22.739076	13.297897	3.0105893	[illegible]
4	8	23.503313	15.923596	2.9303297	[illegible]
4	9	24.664736	19.138121	2.9443376	[illegible]
4	10	26.512913	23.267566	2.8704295	[illegible]

4	11	29.822920	29.160095	2.7457621	1.0098893
4	12	29.823320	29.161633	2.7457391	1.0098843
4	13	29.824691	29.163816	2.7456515	1.0098621
4	14	29.827847	29.168840	2.7454419	1.0098358
5	1	20.242206	4.7577940	3.1568790	1.5708000
5	2	20.268601	6.1179272	3.1660130	1.5320060
5	3	20.360060	7.6439472	3.1705580	1.4859232
5	4	20.540189	9.3718435	3.1690206	1.4440413
5	5	20.843471	11.350189	3.1597942	1.3923710
5	6	21.322725	13.649677	3.1401304	1.3374532
5	7	22.062957	16.376142	3.1062970	1.2744[illegible]
5	8	23.214565	19.707721	3.0523794	1.2024080
5	9	25.075403	23.972962	2.9675440	1.1170664
5	10	28.440739	30.027203	2.8273062	1.0120716
5	11	28.441140	30.027846	2.8272806	1.0120661
5	12	28.442521	30.030055	2.8271890	1.0120438
5	13	28.445695	30.035133	2.8269678	1.0120144
5	14	28.451976	30.045178	2.8265074	1.0119877
6	1	18.854647	6.1453530	3.3172832	1.5708000
6	2	18.890642	7.7367351	3.3214834	1.5255852
6	3	19.017188	9.5412406	3.3186388	1.4760669
6	4	19.271139	11.609406	3.3067724	1.4213620
6	5	19.708741	14.013726	3.2826504	1.3605056
6	6	20.420114	16.861929	3.2419154	1.2921695
6	7	21.563276	20.333926	3.1778418	1.2142781
6	8	23.448045	24.758962	3.0785580	1.1237[illegible]
6	9	26.893233	30.994923	2.9177430	1.0118346

6	10	26.893636	30.995568	2.9177156	1.0118283
6	11	26.895035	30.997805	2.9176179	1.0118[illegible]
6	12	26.898246	31.002937	2.9173814	1.0117735
6	13	26.904586	31.013073	2.9168918	1.0117[illegible]
6	14	26.916515	31.032137	2.9159328	1.0116158
7	1	17.225622	7.7743780	3.5062008	1.5708000
7	2	17.277335	9.6713296	3.5021962	1.5143247
7	3	17.462584	11.850030	3.4871942	1.45[illegible]
7	4	17.843070	14.385086	3.4571868	1.38[illegible]
7	5	18.516811	17.385787	3.4068618	1.3123069
7	6	19.653212	21.032342	3.3293612	1.2262506
7	7	21.578647	25.650983	3.2095872	1.1274226
7	8	25.142193	32.095087	3.0193468	1.0078422
7	9	25.142601	32.095735	3.0193168	1.0078360
7	10	25.144031	32.098001	3.0192102	1.0078168
7	11	25.147303	32.103186	3.0189536	1.0077770
7	12	25.153755	32.113408	3.0184250	1.0077092
7	13	25.165872	32.132602	3.0173932	1.0076036
7	14	25.188122	32.167835	3.0154282	1.0074446
8	1	15.274100	9.7259000	3.7375819	1.5708000
8	2	15.354420	12.044829	3.7192519	1.5016312
8	3	15.648343	14.750144	3.6813391	1.4238467
8	4	16.267577	17.951716	3.6168573	1.3363729
8	5	17.399894	21.826135	3.5162567	1.2379612
8	6	19.395643	26.688101	3.3651757	1.1271290
8	7	23.142100	33.371001	3.1342130	.99740770
8	8	23.142513	33.372632	3.1341908	.[illegible]

8	9	23.143993	33.374924	3.1340618	[illegible]
8	10	23.147373	33.380160	3.1337762	[illegible]
8	11	23.154022	33.390454	3.1331924	.09726220
8	12	23.166484	33.409744	3.1320571	.09714420
8	13	23.189320	33.445083	3.1299049	[illegible]
8	14	23.232639	33.512075	3.1257991	[illegible]
9	1	12.869196	12.130804	4.0397914	1.5904000
9	2	13.009327	15.071225	3.9917714	[illegible]
9	3	13.533823	18.558974	3.9046624	[illegible]
9	4	14.665580	22.755265	3.7658846	1.2475676
9	5	16.790812	27.540070	3.5612392	[illegible]
9	6	20.838443	34.896734	3.2634694	[illegible]
9	7	20.838882	34.897302	3.2634307	[illegible]
9	8	20.840448	34.899693	3.2632929	[illegible]
9	9	20.844015	34.904914	3.2629640	.07400960
9	10	20.851014	34.915275	3.2622963	[illegible]
9	11	20.864099	34.934566	3.2610063	[illegible]
9	12	20.888017	34.969814	3.2585738	[illegible]
9	13	20.933332	35.036555	3.2538989	[illegible]
9	14	21.025757	35.172523	3.2445983	.07340640
10	1	9.7749937	15.225007	4.4702368	[illegible]
10	2	10.075958	19.177489	4.3529706	1.4106640
10	3	11.208034	23.899239	4.1333854	[illegible]
10	4	13.609575	29.544011	3.8163444	[illegible]
10	5	18.214536	36.771689	3.3970292	[illegible]
10	6	18.215020	36.772325	3.3969807	[illegible]
10	7	18.216744	36.774631	3.3968111	[illegible]

10	8	18.220647	36.779841	3.3964118	[illegible]
10	9	18.228311	36.790029	3.3956066	[illegible]
10	10	18.242576	36.809012	3.3940626	[illegible]
10	11	18.266574	36.843364	3.3911712	[illegible]
10	12	18.317752	36.908970	3.3856452	[illegible]
10	13	18.418150	37.042276	3.3744075	[illegible]
10	14	18.659276	37.361246	3.3493707	[illegible]
11	1	5.4741122	19.325888	5.2919271	[illegible]
11	2	6.5210464	25.468238	4.7986020	[illegible]
11	3	9.7635577	31.330418	4.1254873	[illegible]
11	4	15.594681	39.008886	3.4462224	[illegible]
11	5	15.595240	39.009471	3.4461601	[illegible]
11	6	15.597281	39.011605	3.4459376	[illegible]
11	7	15.601803	39.016426	3.4454170	[illegible]
11	8	15.610886	39.025824	3.4443705	[illegible]
11	9	15.627590	39.043277	3.4424051	[illegible]
11	10	15.657026	39.074065	3.4387351	[illegible]
11	11	15.715194	39.134762	3.4317555	[illegible]
11	12	15.832192	39.256930	3.4176982	[illegible]
11	13	16.115087	39.551154	3.3860777	[illegible]
11	14	17.250732	40.509098	3.3014076	[illegible]
12	1	-1.9298951	26.920085	6.7817106	[illegible]

APPENDIX II

THE EVALUATION OF $\int_0^{25} \sigma_n \, r \, dr$ BY NUMERICAL METHODS

For the determination of the indentation stress σ_z in 4.6, Chapter IV, the value of the definite integral $\int_0^{25} \sigma_n \, r \, dr$ was required. This integral was evaluated using the Gauss quadrature formula (National Physical Laboratory [1961]).

By this formula

$$\int_0^{25} \sigma_n \, r \, dr = \int_0^{25} y(r) \, dr$$

$$= \frac{25}{2} \int_{-1}^{1} y(X) \, dX$$

$$= \frac{25}{2} \sum_{r=1}^{n} w_r^{(n)} \, y(X_r^{(n)}) ,$$

where $y(r) = \sigma_n \, r$, $X = \frac{2r}{25} - 1$, $X_r^{(n)}$ are zeros of Legendre polynomials and $w_r^{(n)}$ are weights. Values of $X_r^{(n)}$ and $w_r^{(n)}$ corresponding to $n = 16$ that were used in this formula are listed as follows:

$X_r^{(n)}$	w_r^n
-0.98940	0.02715
-0.94458	0.06225
-0.86563	0.09516
-0.75540	0.12463
-0.61788	0.14960
-0.45802	0.16916
-0.28160	0.18260

(continued)

$X_r^{(n)}$	w_r^n
-0.09501	0.18945
0.09501	0.18945
0.28160	0.18260
0.45802	0.16916
0.61788	0.14960
0.75540	0.12463
0.86563	0.09516
0.94458	0.06225
0.98940	0.02715

Also, the values of r, σ_n, X and $y(X)$ for 22 points along AE (Fig. 3) are given as follows where the value of unity has been assigned to k:

r	σ_n	X	$y(X)$
0	11.210	-1.00000	0
0.5926	8.8907	-0.95235	5.2953
2.2422	7.4639	-0.82052	16.7356
3.9224	6.7594	-0.68621	26.5131
5.3884	6.3349	-0.56893	34.1350
6.0957	6.1637	-0.51234	37.5721
7.3707	5.8955	-0.41034	43.4540
8.7000	5.6552	-0.30400	49.2002
9.7621	5.4831	-0.21903	53.5266
11.2590	5.2608	-0.09928	59.2313
12.8661	5.0405	0.02929	64.8516
14.2193	4.8742	0.13754	69.3077
15.2719	4.7380	0.22175	72.3583
15.9578	4.6524	0.27662	74.2421

(continued)

r	σ_n	X	y(X)
17.2256	4.5062	0.37805	77.6220
18.8546	4.3173	0.50837	81.4010
20.2422	4.1569	0.61938	84.1448
21.4417	4.0165	0.71534	86.1206
22.4942	3.8907	0.79954	87.5182
23.4250	3.7761	0.87400	88.4551
24.2551	3.6701	0.94041	89.0186
25.0000	3.5708	1.00000	89.2700

Since no values of X listed above coincided with any value of $X_r^{(n)}$, it was necessary to interpolate the values of $y(X_r^{(n)})$. This was accomplished using the Lagrange's method of interpolation with unequal intervals (National Physical Laboratory [1961]) using 4 consecutive values of $y(X)$ in any one interpolation calculation. The Lagrange's method thus used may be written as

$$y(X_r^{(n)}) = L(X_{-1})y(X_{-1})+L(X_1)y(X_1)+L(X_2)y(X_2)+L(X_3)y(X_3)$$

or any variation thereof depending upon the value of $X_r^{(n)}$ used in relation to the 4 values of X. The L's are the Lagrange coefficients and extensive tables for them are available only for equal interval interpolation. However, since the values of X form unequal intervals, this necessitated the computation of new Lagrange coefficients by a simple technique outlined by Comrie [1959]. In way of an illustration, the calculation of $y(X_r')$ is given here; the values and not the calculations of the other $y(X_r^{(n)})$ being given later.

$X_r^1 = 0.9840$

X	y(X)	DIFF.	NUM.x10^6	DEN.x10^6	L(X)	L(X)y(X)
$X_1 = -1.00000$	0	0.01060	1,897.06	2,683.60	+0.706909	0
$X_2 = -0.95235$	5,2953	0.03705	542.75	1,671.81	+0.324648	+1.7191
$X_3 = -0.82052$	16.7356	0.16888	119.07	3,177.89	-0.037468	-0.6270
$X_4 = -0.68621$	26.5131	0.30319	66.32	11,216.51	+0.005913	+0.1568

$y(X_r') = +1.24089$

The values of the columns not self-explanatory were evaluated under the following scheme as given by Comrie.

$X_r^1 = n$

X	DIFF.	NUM.	DEN.	$L(X) = \frac{\text{NUM.}}{\text{DEN.}}$
$X_1 = a$	n-a	(b-n)(c-n)(d-n)	(b-a)(c-a)(d-a)	+
$X_2 = b$	b-n	(n-a)(c-n)(d-n)	(a-b)(c-b)(d-b)	+
$X_3 = c$	c-n	(n-a)(b-n)(d-n)	(a-c)(t-c)(d-c)	-
$X_4 = d$	d-n	(n-a)(b-n)(c-n)	(a-d)(b-d)(c-d)	+

In this example n lies between a and b, but the method is applicable for other positions of n. The sign of L(X) is determined by assigning + to the two adjacent values of n and then alternating the sign in each direction.

The remaining values of $y(X_r^{(n)})$ are included in the following tabulation from which the definite integral under consideration was evaluated.

$y(X_r^{(n)})$	$w_r^{(n)}$	$w_r^{(n)}\ y(X_r^{(n)})$
1.2489	0.02715	0.03391
6.0872	0.06225	0.37895
13.0659	0.09516	1.24334
21.6341	0.12463	2.69621
31.0420	0.14960	4.64376
40.7511	0.16916	6.89333
50.3632	0.18260	9.19647
59.4181	0.18945	11.25682
67.6350	0.18945	12.81352
74.4173	0.18260	13.58882
80.0079	0.16916	13.53390
84.1110	0.14960	12.58267
86.8268	0.12463	10.82105
88.3745	0.09516	8.40963
89.0452	0.06225	5.54342
89.2444	0.02715	2.42316

$$\sum w_r^{(n)}\ y(X_r^{(n)}) = 116.05896$$

$$\int_0^{25} y(r)\ dt = \frac{25}{2} \sum w_r^{(n)}\ y(X_r^{(n)}) = 1450.73700$$

Therefore $$\sigma_z = \frac{2k}{625} \int_0^{25} y(r)\ dr$$

$$= 4.6424\ k\ .$$

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